Anna Futa and Dariusz Partyka

THE SCHWARZ TYPE INEQUALITY FOR HARMONIC FUNCTIONS OF THE UNIT DISC SATISFYING A SECTORIAL CONDITION

Summary

Let $T_1$, $T_2$ and $T_3$ be closed arcs contained in the unit circle $T$ with the same length $2\pi/3$ and covering $T$. In the paper [3] D. Partyka and J. Zając obtained the sharp estimation of the module $|F(z)|$ for $z \in \mathbb{D}$ where $\mathbb{D}$ is the unit disc and $F$ is a complex-valued harmonic function of $\mathbb{D}$ into itself satisfying the following sectorial condition: For each $k \in \{1, 2, 3\}$ and for almost every $z \in T_k$ the radial limit of the function $F$ at the point $z$ belongs to the angular sector determined by the convex hull spanned by the origin and arc $T_k$. In this article a more general situation is considered where the three arcs are replaced by a finite collection $T_1, T_2, \ldots, T_n$ of closed arcs contained in $T$ with positive length, total length $2\pi$ and covering $T$.

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1. Introduction

Throughout the paper we always assume that all topological notions and operations are understood in the complex plane $E(\mathbb{C}) := (\mathbb{C}, \rho_e)$, where $\rho_e$ is the standard euclidean metric. We will use the notations $\text{cl}(A)$ and $\text{fr}(A)$ for the closure and boundary of a set $A \subset \mathbb{C}$ in $E(\mathbb{C})$, respectively. By $\text{Har}(\Omega)$ we denote the class of all complex-valued harmonic functions in a domain $\Omega$, i.e., the class of all twice
continuously differentiable functions $F$ in $\Omega$ satisfying the Laplace equation
\[
\frac{\partial^2 F(z)}{\partial x^2} + \frac{\partial^2 F(z)}{\partial y^2} = 0, \quad z = x + iy \in \Omega.
\]
The sets $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ are the unit disc and unit circle, respectively. The standard measure of a Lebesgue measurable set $A \subset \mathbb{T}$ will be denoted by $|A|_1$. In particular, if $A$ is an arc then $|A|_1$ means its length. Set $\mathbb{Z}_{p,q} := \{k \in \mathbb{Z} : p \leq k \leq q\}$ for any $p,q \in \mathbb{Z}$.

**Definition 1.1.** For every $n \in \mathbb{N}$ a sequence $Z_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ is said to be a partition of the unit circle provided $T_k$ is a closed arc of length $|T_k|_1 > 0$ for $k \in Z_{1,n}$ as well as
\[
\bigcup_{k=1}^n T_k = \mathbb{T} \quad \text{and} \quad \sum_{k=1}^n |T_k|_1 = 2\pi. \tag{1.1}
\]

For any function $F : \mathbb{D} \to \mathbb{C}$ and $z \in \mathbb{T}$ we define the set $F^{**}(z)$ of all $w \in \mathbb{C}$ such that there exists a sequence $\mathbb{N} \ni n \mapsto r_n \in [0; 1)$ satisfying the equalities
\[
\lim_{n \to +\infty} r_n = 1 \quad \text{and} \quad \lim_{n \to +\infty} F(r_n z) = w.
\]

**Definition 1.2.** By the sectorial boundary normalization given by a partition $Z_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ of the unit circle we mean the class $\mathcal{N}(T_1, T_2, \ldots, T_n)$ of all functions $F : \mathbb{D} \to \mathbb{D}$ such that for every $k \in Z_{1,n}$ and almost every (a.e. in abbr.) $z \in T_k$,
\[
F^{**}(z) \subset D_k := \{ru : 0 \leq r \leq 1, u \in T_k\} = \text{conv}(T_k \cup \{0\}). \tag{1.2}
\]

Given $n \in \mathbb{N}$ and a partition $Z_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ of the unit circle we will study the Schwarz type inequality for the class
\[
\mathcal{F} := \text{Har}(\mathbb{D}) \cap \mathcal{N}(T_1, T_2, \ldots, T_n).
\]

If $n \leq 2$ then we have a trivial sharp estimation $|F(z)| \leq 1$ for $F \in \mathcal{F}$ and $z \in \mathbb{D}$, where the equality is attained for a constant function. Therefore, from now on we always assume that $n \geq 3$.

In Section 2 we prove a few useful properties of the class $\mathcal{F}$. Most essential here is Theorem 2.3. We use it to show in Section 3 Theorem 3.1, which is our main result. Then we apply the last theorem in specific cases; cf. Examples 3.4 and 3.5. In particular, we derive the estimation (3.13), obtained by D. Partyka and J. Zajac in [3, Corollary 2.2]. Thus the estimation (3.1), valid for an arbitrary partition of $\mathbb{T}$, generalizes the one (3.13), which holds only in the case where $n = 3$ and the arcs $T_1$, $T_2$ and $T_3$ have the same length. Note that the estimation (3.12) is a directional improvement of the radial one (3.13). In Example 3.5 we study a general case of an arbitrary partition of the unit circle. As a result, we derive reasonable estimations (3.23) and (3.24), which depend on the largest length among the ones $|T_k|_1$ for $k \in Z_{1,n}$.
2. Auxiliary results

Let \( P[f] \) stand for the Poisson integral of an integrable function \( f : \mathbb{T} \to \mathbb{C} \), i.e.,
\[
\begin{align*}
P[f](z) &:= \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \frac{1 - |z|^2}{|u - z|^2} |du| = \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \text{Re} \frac{u + z}{u - z} |du|, \quad z \in \mathbb{D}.
\end{align*}
\]
(2.1)

The Poisson integral provides the unique solution to the Dirichlet problem in the unit disc \( \mathbb{D} \) provided that the boundary function \( f \) is continuous. It means that \( P[f] \) is a harmonic function in \( \mathbb{D} \), which has a continuous extension to the closed disc \( \text{cl}(\mathbb{D}) \) and its boundary values function coincides with \( f \). For any function \( F : \mathbb{D} \to \mathbb{C} \) we define the radial limit function of \( F \) by the formula
\[
T \ni z \mapsto F^*(z) := \begin{cases} 
\lim_{r \to 1^-} F(rz), & \text{if the limit exists,} \\
0, & \text{otherwise.} 
\end{cases}
\]
(2.2)

Since a real-valued harmonic and bounded function in \( \mathbb{D} \) has the radial limit for a.e. point of \( \mathbb{T} \) (see e.g. [2, Cor. 1, Sect. 1.2]), it follows that \( F^* = (\text{Re} F)^* + i(\text{Im} F)^* \) almost everywhere on \( \mathbb{T} \) provided \( F \in \text{Har}(\mathbb{D}) \) is bounded in \( \mathbb{D} \). Therefore,
\[
F^{**}(z) = \{F^*(z)\} \quad \text{for every } F \in \mathcal{F} \text{ and a.e. } z \in \mathbb{T}.
\]
(2.3)

In particular, for each function \( F : \mathbb{D} \to \mathbb{D} \), \( F \in \mathcal{F} \) if and only if \( F \in \text{Har}(\mathbb{D}) \) and \( F^*(z) \in D_k \) for \( k \in \mathbb{Z}_{1,n} \) and a.e. \( z \in T_k \). From the property (2.2) it follows that for each \( F \in \mathcal{F} \) the sequence \( \mathbb{N} \ni m \mapsto f_m \), where
\[
T \ni u \mapsto f_m(u) := F((1 - \frac{1}{m})u), \quad m \in \mathbb{N},
\]
is convergent to \( F^* \) almost everywhere on \( \mathbb{T} \). Then applying the dominated convergence theorem we see that for every \( z \in \mathbb{D} \),
\[
F((1 - \frac{1}{m})z) = P[f_m](z) \to P[F^*](z) \quad \text{as } m \to +\infty,
\]
which yields
\[
F = P[F^*], \quad F \in \mathcal{F}.
\]
(2.4)

Let \( \chi_I \) be the characteristic function of a set \( I \in \mathbb{T} \), i.e., \( \chi_I(t) := 1 \) for \( t \in I \) and \( \chi_I(t) := 0 \) for \( t \in \mathbb{T} \setminus I \).

**Lemma 2.1.** For all \( F \in \mathcal{F} \) and \( z \in \mathbb{D} \) there exists a sequence \( \mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k \) such that the following equality holds
\[
F(z) = \sum_{k=1}^{n} c_k P[\chi_{T_k}](z).
\]
(2.5)

**Proof.** Fix \( F \in \mathcal{F} \) and \( z \in \mathbb{D} \). Since \( |T_k| > 0 \) for \( k \in \mathbb{Z}_{1,n} \), it follows that
\[
0 < p_k := P[\chi_{T_k}](z) < 1, \quad k \in \mathbb{Z}_{1,n}.
\]
By (1.2) each sector $D_k$, $k \in \mathbb{Z}_{1,n}$, is closed and convex. Moreover, from (1.2) and (2.2) we see that $F^*(z) \in D_k$ for $k \in \mathbb{Z}_{1,n}$ and a.e. $z \in T_k$. Then applying the integral mean value theorem for complex-valued functions we deduce from (2.5) that
\[
c_k := P\left[ \frac{1}{p_k} \cdot F^* \cdot \chi_{T_k} \right](z) \in D_k, \quad k \in \mathbb{Z}_{1,n}.
\]
Hence and by (2.3),
\[
F(z) = P[F^*](z) = P\left[ \sum_{k=1}^{n} F^* \cdot \chi_{T_k} \right](z) = \sum_{k=1}^{n} P\left[ F^* \cdot \chi_{T_k} \right](z) = \sum_{k=1}^{n} p_k c_k,
\]
which implies the equality (2.4).

\[\square\]

**Lemma 2.2.** For every sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$,
\[
F := \sum_{k=1}^{n} c_k P[\chi_{T_k}] \in \mathcal{F}.
\] (2.6)

**Proof.** Given a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ consider the function $F$ defined by the formula (2.6). Since $P[\chi_{T_k}] \in \text{Har}(\mathbb{D})$ for $k \in \mathbb{Z}_{1,n}$, we see that $F \in \text{Har}(\mathbb{D})$. Furthermore, for each $z \in \mathbb{D}$,
\[
\sum_{k=1}^{n} P[\chi_{T_k}](z) = P\left[ \sum_{k=1}^{n} \chi_{T_k} \right](z) = P[\chi_{\mathbb{T}}](z) = 1,
\]
whence
\[
|F(z)| \leq \sum_{k=1}^{n} |c_k| P[\chi_{T_k}](z) \leq \sum_{k=1}^{n} P[\chi_{T_k}](z) = 1.
\]

By the definition of the function $F$ we have
\[
F^*(z) = \sum_{k=1}^{n} c_k \chi_{T_k}(z), \quad z \in \mathbb{T} \setminus E,
\] (2.7)
where $E$ is the set of all $u \in \mathbb{T}$ such that $u$ is an endpoint of a certain arc among the arcs $T_k$ for $k \in \mathbb{Z}_{1,n}$.

Assume that $|F(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. By the maximum modulus principle for complex-valued harmonic functions (cf. [1, Corollary 1.11, p. 8]) there exists $w \in \mathbb{T}$ such that $F(z) = w$ for $z \in \mathbb{D}$, and so $F^*(z) = w$ for $z \in \mathbb{T}$. By (2.7), $F^*(z) = c_k$ for $k \in \mathbb{Z}_{1,n}$ and $z \in T_k \setminus E$. Therefore $w = c_k \in D_k$ for $k \in \mathbb{Z}_{1,n}$, and so $w \in D_1 \cap D_2 \cap D_3 = \{0\}$. Hence $w = 0$, which contradicts the equality $|w| = 1$. Thus $F(z) < 1$ for $z \in \mathbb{D}$, and so $F : \mathbb{D} \to \mathbb{D}$. Furthermore, from (2.7) it follows that for all $k \in \mathbb{Z}_{1,n}$ and $z \in T_k \setminus E$, $F^*(z) = c_k \in D_k$. Thus $F \in \mathcal{N}(T_1, T_2, \ldots, T_n)$, which implies (2.6).

\[\square\]
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Theorem 2.3. For every compact set $K \subset \mathbb{D}$ there exist a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ and $z_K \in \text{fr}(K)$ such that

$$F_K := \sum_{k=1}^{n} c_k P[X_{T_k}] \in \mathcal{F}$$

and

$$|F(z)| \leq |F_K(z_K)| = \left| \sum_{k=1}^{n} c_k P[X_{T_k}](z_K) \right|, \quad F \in \mathcal{F}, \; z \in K. \quad (2.8)$$

In particular,

$$\max(\{ |F(z)| : F \in \mathcal{F}, \; z \in K \}) = |F_K(z_K)|. \quad (2.9)$$

Proof. Fix a compact set $K \subset \mathbb{D}$. Since $F(K) \subset F(\mathbb{D}) \subset \mathbb{D}$ for $F \in \mathcal{F}$,

$$M_K := \sup(\{ |F(z)| : F \in \mathcal{F}, \; z \in K \}) \leq 1. \quad (2.10)$$

Hence, there exist sequences $\mathbb{N} \ni m \mapsto F_m \in \mathcal{F}$ and $\mathbb{N} \ni m \mapsto z_m \in K$ such that

$$\lim_{m \to +\infty} |F_m(z_m)| = M_K. \quad (2.11)$$

From Lemma 2.1 it follows that for each $m \in \mathbb{N}$ there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_{m,k} \in D_k$ such that

$$F_m(z_m) = \sum_{k=1}^{n} c_{m,k} P[X_{T_k}](z_m). \quad (2.12)$$

Since the set $D_k$ is compact for $k \in \mathbb{Z}_{1,n}$ we see, using the standard technique of choosing a convergent subsequence from a sequence in a compact set, that there exists an increasing sequence $\mathbb{N} \ni l \mapsto m_l \in \mathbb{N}$, a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ and $z'_K \in K$ such that

$$c_{m_l,k} \to c_k \quad \text{as } l \to +\infty \quad \text{for } k \in \mathbb{Z}_{1,n} \quad (2.13)$$

and

$$z_{m_l} \to z'_K \quad \text{as } l \to +\infty. \quad (2.14)$$

By Lemma 2.2, the property (2.8) holds. From (2.13) we conclude that for every $m \in \mathbb{N}$,

$$|F_K(z_m) - F_m(z_m)| = \left| \sum_{k=1}^{n} c_k P[X_{T_k}](z_m) - \sum_{k=1}^{n} c_{m,k} P[X_{T_k}](z_m) \right| \leq \sum_{k=1}^{n} |c_k - c_{m,k}| P[X_{T_k}](z_m) \leq \sum_{k=1}^{n} |c_k - c_{m,k}|,$$

which together with (2.14) leads to

$$\lim_{l \to +\infty} |F_K(z_{m_l}) - F_{m_l}(z_{m_l})| = 0. \quad (2.15)$$
Since $|c_k| \leq 1$ for $k \in \mathbb{Z}_{1,n}$, it follows that
\[
|F_K(z'_K) - F_K(z_m)| \leq \left| \sum_{k=1}^{n} c_k P[\chi_{T_k}](z'_K) - \sum_{k=1}^{n} c_k P[\chi_{T_k}](z_m) \right|
\leq \sum_{k=1}^{n} |c_k| \cdot |P[\chi_{T_k}](z'_K) - P[\chi_{T_k}](z_m)|
\leq \sum_{k=1}^{n} |P[\chi_{T_k}](z'_K) - P[\chi_{T_k}](z_m)|,
\quad m \in \mathbb{N}.
\]
This together with (2.15) yields
\[
\lim_{l \to +\infty} |F_K(z'_K) - F_K(z_{ml})| = 0.
\] (2.17)
Since for every $l \in \mathbb{N}$,
\[
|F_K(z'_K) - F_{ml}(z_{ml})| \leq |F_K(z'_K) - F_K(z_{ml})| + |F_K(z_{ml}) - F_{ml}(z_{ml})|,
\]
we deduce from (2.17) and (2.16) that
\[
\lim_{l \to +\infty} |F_{ml}(z_{ml})| = |F_K(z'_K)|.
\]
Hence and by (2.12), $|F_K(z'_K)| = M_K$. Since $F_K \in \text{Har}(\mathbb{D})$, the maximum modulus principle for complex-valued harmonic function (cf. [1, Corollary 1.11, p. 8]) implies that there exists $z_K \in \text{fr}(K)$ such that $|F_K(z)| \leq |F_K(z_K)|$ for $z \in K$. In particular, $M_K = |F_K(z'_K)| \leq |F_K(z_K)|$. On the other hand, by (2.8) and (2.11), $|F_K(z_K)| \leq M_K$. Eventually, $|F_K(z_K)| = M_K$. This implies (2.10), and thereby, the inequality (2.9) holds, which is the desired conclusion. \[\square\]

3. Estimations

As an application of Theorem 2.3 we shall prove the following result.

**Theorem 3.1.** For every $z \in \mathbb{D}$ the following inequality holds
\[
|F(z)| \leq 1 - (n - S)p(z), \quad F \in \mathcal{F},
\] (3.1)
where
\[
S := \sup \left\{ \text{Re} \left( \sum_{k=1}^{n} u_k v_k \right) : u \in \mathbb{T}, \; \mathbb{Z}_{1,n} \ni k \mapsto v_k \in D_k \right\}
\] (3.2)
and
\[
p(z) := \min(\{P[\chi_{T_k}](z) : k \in \mathbb{Z}_{1,n}\}).
\] (3.3)

**Proof.** It is clear that $K := \{z\}$ is a compact set for a given $z \in \mathbb{D}$. By Theorem 2.3 there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ such that
\[
F_K := \sum_{k=1}^{n} c_k P[\chi_{T_k}] \in \mathcal{F}
\]
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|F(z)| ≤ |FK(z)|, \quad F ∈ F. \tag{3.4}

Setting \( u := \frac{FK(z)}{|FK(z)|} \) if \( FK(z) \neq 0 \) and \( u := 1 \) if \( FK(z) = 0 \), we see that \( u ∈ T \) and \( FK(z) = u|FK(z)| \). Hence

\[
|FK(z)| = \overline{u}FK(z) = \text{Re}(\overline{u}FK(z)) = \sum_{k=1}^{n} \text{Re}(\overline{uc_k}p_k) = \sum_{k=1}^{n} \text{Re}(\overline{uc_k})p_k, \tag{3.5}
\]

where \( p_k := P[\chi_{Tk}](z) \) for \( k ∈ Z_{1,n} \). Since

\[
\sum_{k=1}^{n} p_k = 1 \quad \text{and} \quad \text{Re}(\overline{uc_k}) \leq M := \max(\{\text{Re}(\overline{uc_l}) : l ∈ Z_{1,n}\}) \leq 1, \quad k ∈ Z_{1,n},
\]

we deduce from the formula (3.3) that

\[
\sum_{k=1}^{n} \text{Re}(\overline{uc_k})p_k = \sum_{k=1}^{n} (\text{Re}(\overline{uc_k}) - M + M)p_k
= M \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} (\text{Re}(\overline{uc_k}) - M)p_k
\leq M \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} (\text{Re}(\overline{uc_k}) - M)p(z)
= M \sum_{k=1}^{n} (p_k - p(z)) + p(z) \sum_{k=1}^{n} \text{Re}(\overline{uc_k})
\leq \sum_{k=1}^{n} (p_k - p(z)) + p(z) \sum_{k=1}^{n} \text{Re}(\overline{uc_k})
= 1 - np(z) + p(z) \sum_{k=1}^{n} \text{Re}(\overline{uc_k}).
\]

This together with (3.5) and (3.2) yields

\[
|FK(z)| \leq 1 - np(z) + p(z) \sum_{k=1}^{n} \text{Re}(\overline{uc_k})
\leq 1 - np(z) + p(z)S
= 1 - (n - S)p(z).
\]

Hence and by (3.4) we obtain the estimation (3.1), which proves the theorem. \( □ \)

The estimation (3.1) is useful provided we can estimate \( p(z) \) from below and \( S \) from above. The first task is easy and depends on the following quantity

\[
\delta := \frac{1}{2} \min(\{|Tk|_1 : k ∈ Z_{1,n}\}). \tag{3.6}
\]
Lemma 3.2. For every \( \alpha \in (0; \pi/2] \) the following estimation holds
\[
P[\chi_{I_\alpha}](z) \geq P[\chi_{I_\alpha}](|z|) = \frac{2}{\pi} \arctan \left( \frac{\sin(\alpha)}{|z| + \cos(\alpha)} \right) - \frac{\alpha}{\pi}, \quad z \in \mathbb{D},
\] (3.7)
where \( I_\alpha := \{e^{it} : |t - \pi| \leq \alpha \} \).

Proof. Given \( \{C_I\} \) are the endpoints of the arc \( I_\alpha \). Consequently, \( \log(z) \) is holomorphic and
\[\frac{d}{dt} \log(z - e^{it}) = \frac{ie^{it}}{e^{it} - z}, \quad t \in [\pi - \alpha; \pi + \alpha].\]
Here we understand the function \( \log \) as the inverse of the function \( \exp|_{\Omega} \), where \( \Omega := \{\zeta \in \mathbb{C} : |\text{Im}\zeta| < \pi\} \). By (2.1) we have
\[
P[\chi_{I_\alpha}](z) = \frac{1}{2\pi} \int_0^{\pi} \chi_{I_\alpha}(u) \text{Re} \frac{u + z}{u - z} |du| \]
\[= \frac{1}{2\pi} \int_{\pi - \alpha}^{\pi + \alpha} \text{Re} \frac{e^{it} + z}{e^{it} - z} dt \]
\[= \frac{1}{2\pi} \int_{\pi - \alpha}^{\pi + \alpha} \text{Re} \left( \frac{2e^{it}}{e^{it} - z} - 1 \right) dt \]
\[= \frac{1}{\pi} \int_{\pi - \alpha}^{\pi + \alpha} \text{Im} \left( \frac{ie^{it}}{e^{it} - z} \right) dt - \frac{\alpha}{\pi} \]
\[= \frac{1}{\pi} \int_{\pi - \alpha}^{\pi + \alpha} \text{Im} \frac{d}{dt} \log(z - e^{it}) dt - \frac{\alpha}{\pi} \]
\[= \frac{1}{\pi} \int_{\pi - \alpha}^{\pi + \alpha} \log(z - e^{it}) dt - \frac{\alpha}{\pi} \]
\[= \frac{1}{\pi} \log(z - e^{i\theta}) - \frac{\alpha}{\pi}, \quad \theta \in \mathbb{R}.
\] (3.8)

Consequently,
\[
\frac{d}{d\theta} P[\chi_{I_\alpha}](re^{i\theta}) = \frac{1}{\pi} \text{Im} \left[ \frac{ire^{i\theta}}{re^{i\theta} + e^{i\alpha}} - \frac{ire^{i\theta}}{re^{i\theta} + e^{-i\alpha}} \right]
\]
\[= \frac{r}{\pi} \text{Im} \left[ \frac{e^{i\theta}(-e^{i\alpha} + e^{-i\alpha})}{(re^{i\theta} + e^{i\alpha})(re^{i\theta} + e^{-i\alpha})} \right]
\]
\[= \frac{2r \sin(\alpha)}{\pi} \text{Im} \left[ \frac{e^{i\theta}(re^{-i\theta} + e^{-i\alpha})(re^{-i\theta} + e^{i\alpha})}{|re^{i\theta} + e^{i\alpha}|^2|re^{i\theta} + e^{-i\alpha}|^2} \right]
\]
\[= \frac{2r \sin(\alpha)}{\pi} \text{Im} \left[ \frac{r^2e^{-i\theta} + re^{-i\alpha} + re^{i\alpha} + e^{i\theta}}{|re^{i\theta} + e^{i\alpha}|^2|re^{i\theta} + e^{-i\alpha}|^2} \right]
\]
\[= \frac{2r(1 - r^2)\sin(\alpha)\sin(\theta)}{\pi|re^{i\theta} + e^{i\alpha}|^2|re^{i\theta} + e^{-i\alpha}|^2}, \quad \theta \in \mathbb{R}.
\]
Combining this with (3.8) we derive the estimation (3.7), which proves the lemma. □

**Corollary 3.3.** The following estimation holds

\[ p(z) \geq P[X_{I_\delta}(|z|)] = \frac{2}{\pi} \arctan \left( \frac{\sin(\delta)}{|z| + \cos(\delta)} \right) - \frac{\delta}{\pi}, \quad z \in \mathbb{D}, \]  

(3.9)

where \( p(z) \) and \( \delta \) are defined by the formulas (3.3) and (3.6), respectively.

**Proof.** Let \( \mathbb{Z}_{1,n} \ni k \mapsto a_k \in \mathbb{T} \) be the sequence of midpoints of the partition \( \mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T} \), i.e.,

\[ T_k := \{ a_k e^{it} : |t| \leq \alpha_k \}, \]  

(3.10)

where \( \alpha_k := \frac{1}{2} |T_k|_1 \) for \( k \in \mathbb{Z}_{1,n} \). Hence and by (3.6) we obtain \( I_\delta \subset I_{\alpha_k} \) for \( k \in \mathbb{Z}_{1,n} \), where \( I_\alpha := \{ e^{it} : |t - \pi| \leq \alpha \} \) for \( \alpha \in (0; \pi] \). Then applying the formula (2.1) we see that for an arbitrarily fixed \( z \in \mathbb{D} \),

\[ P[X_{I_{\alpha_k}}(|z|)] = P[X_{I_\delta}(|z|)] + P[X_{I_{\alpha_k \setminus I_\delta}}(|z|)] \geq P[X_{I_\delta}(|z|)], \quad k \in \mathbb{Z}_{1,n}. \]

Therefore

\[ \min(\{ P[X_{I_{\alpha_k}}(|z|) : k \in \mathbb{Z}_{1,n} \}) = P[X_{I_\delta}(|z|)], \]  

(3.11)

because \( \delta = \alpha_{k'} \) for some \( k' \in \mathbb{Z}_{1,n} \). Fix \( k \in \mathbb{Z}_{1,n} \). Using the rotation mapping \( \mathbb{C} \ni \zeta \mapsto \varphi(\zeta) := -a_k^{-1} \zeta \) we have \( \varphi(T_k) = I_{\alpha_k} \). Then integrating by substitution we deduce from the formula (2.1) that

\[ P[X_{T_k}(z)] = P[X_{\varphi(T_k)}](\varphi(z)) = P[X_{I_{\alpha_k}}](\varphi(z)). \]

On the other hand, by Lemma 3.2,

\[ P[X_{I_{\alpha_k}}](\varphi(z)) \geq P[X_{I_{\alpha_k}}](|\varphi(z)|) = P[X_{I_{\alpha_k}}](|z|). \]

Thus

\[ P[X_{T_k}(z)] \geq P[X_{I_{\alpha_k}}](|z|), \quad k \in \mathbb{Z}_{1,n}. \]

Combining this with (3.3) and (3.11) we derive the estimation (3.9), which completes the proof. □

A more difficult problem is to estimate from above the quantity \( S \) given by the formula (3.2). It will be studied elsewhere. Now we present two examples.

**Example 3.4.** Suppose that \( \mathbb{Z}_{1,3} \ni k \mapsto T_k \subset \mathbb{T} \) is a partition of \( \mathbb{T} \) such that \( |T_1|_1 = |T_2|_1 = |T_3|_1 \). As in the proof of [3, Theorem 2.1] we can show that \( S \leq 2 \). Hence and by Theorem 3.1 we obtain

\[ |F(z)| \leq 1 - p(z) = 1 - \min(\{ P[X_{T_k}(z) : k \in \mathbb{Z}_{1,3}) \}), \quad F \in \mathcal{F}, \ z \in \mathbb{D}. \]  

(3.12)

Corollary 3.3 now implies the estimation

\[ |F(z)| \leq \frac{4}{3} - \frac{2}{\pi} \arctan \left( \frac{\sqrt{3}}{1 + 2|z|} \right), \quad F \in \mathcal{F}, \ z \in \mathbb{D}; \]  

(3.13)

cf. [3, Corollary 2.2]. Therefore, the estimation (3.12) is a directional type enhancement of the radial one (3.13) for the class \( \mathcal{F} \).
Example 3.5. Suppose that \( Z_{1,n} \ni k \mapsto T_k \subset T \) is a partition of \( T \) such that
\[
\Delta := \max\{ |T_k|_1 : k \in Z_{1,n} \} \leq \frac{\pi}{2}.
\] (3.14)
Then
\[
N := \text{Ent}\left( \frac{\pi}{2\Delta} \right) \geq 1.
\] (3.15)
Fix \( u \in T \) and a sequence \( Z_{1,n} \ni k \mapsto v_k \in D_k \). There exist a bijective function \( \sigma \) of the set \( Z_{1,n} \) onto itself and an increasing sequence \( Z_{1,n} \ni k \mapsto \alpha_k \in \mathbb{R} \) such that \( \alpha_n = 2\pi + \alpha_0, u \in T_{\sigma(1)} \) and
\[
T_{\sigma(k)} = \{ e^{it} : \alpha_{k-1} \leq t \leq \alpha_k \}, \quad k \in Z_{1,n}.
\]
Hence there exist \( \theta \in [\alpha_0;\alpha_1] \) and a sequence \( Z_{1,n} \ni k \mapsto (r_k e^{i\theta_k}) \in [0; 1] \times \mathbb{R} \) such that \( u = e^{i\theta}, v_k = r_k e^{i\theta_k} \) for \( k \in Z_{1,n} \) and
\[
\alpha_{k-1} \leq \theta_{\sigma(k)} \leq \alpha_k, \quad k \in Z_{1,n}.
\] (3.16)
Since for each \( k \in Z_{1,n} \),
\[
\text{Re}(\overline{uv}_k) = \text{Re}\left( r_k e^{i\theta_k} e^{-i\theta} \right) = \text{Re}\left( r_k e^{i(\theta_k - \theta)} \right) = r_k \cos(\theta_k - \theta),
\]
we conclude that
\[
\text{Re}(\overline{uv}_k) \leq \max\{ 0, \cos(\theta_k - \theta) \}, \quad k \in Z_{1,n}.
\] (3.17)
From (3.14) it follows that
\[
\alpha_j - \alpha_i = \sum_{l=i+1}^{j} (\alpha_l - \alpha_{l-1}) \leq (j-i)\Delta, \quad i, j \in Z_{0,n}, i < j.
\] (3.18)
Setting
\[
p := \min\{ k \in Z_{1,n} : \alpha_k \geq \frac{\pi}{2} + \theta \} \quad \text{and} \quad q := \max\{ k \in Z_{1,n} : \alpha_k < \frac{3\pi}{2} + \theta \}
\]
we conclude from (3.15) and (3.18) that
\[
N\Delta \leq \frac{\pi}{2} \leq \alpha_p - \theta \leq \alpha_p - \alpha_0 \leq p\Delta
\]
as well as
\[
N\Delta \leq \frac{\pi}{2} = \alpha_q + \frac{\pi}{2} - \alpha_q < 2\pi + \theta - \alpha_q \leq \alpha_n - \alpha_q + \alpha_1 - \alpha_0 \leq (n-q+1)\Delta.
\]
Therefore \( N \leq p \) and \( q + N \leq n \). Given \( k \in Z_{1,n} \) the following four cases can appear. If \( p+1-N \leq k \leq p \) then by (3.16) and (3.18),
\[
\frac{\pi}{2} + \theta - \theta_{\sigma(k)} \leq \alpha_p - \alpha_{k-1} \leq (p+1-k)\Delta \leq N\Delta \leq \frac{\pi}{2}
\]
as well as
\[
\frac{\pi}{2} + \theta - \theta_{\sigma(k)} > \alpha_{p-1} - \alpha_p \geq -\Delta \geq -\frac{\pi}{2},
\]
which gives
\[ \cos(\theta_{\sigma(k)} - \theta) = \sin(\pi/2 + \theta - \theta_{\sigma(k)}) \leq \sin((p + 1 - k)\Delta). \]

Hence and by (3.17) we obtain
\[ \Re(\overline{u}v_{\sigma(k)}) \leq \sin((p + 1 - k)\Delta), \quad k \in \mathbb{Z}_{p+1-N,p}. \] (3.19)

If \( p + 1 \leq k \leq q \) then by (3.16),
\[ \frac{\pi}{2} + \theta \leq \alpha_{k-1} \leq \theta_{\sigma(k)} \leq \alpha_k < \frac{3\pi}{2} + \theta, \]
and so \( \cos(\theta_{\sigma(k)} - \theta) \leq 0 \). This together with (3.17) leads to
\[ \Re(\overline{u}v_{\sigma(k)}) \leq 0, \quad k \in \mathbb{Z}_{p+1,q}. \] (3.20)

If \( q + 1 \leq k \leq q + N \) then by (3.16) and (3.18),
\[ \theta_{\sigma(k)} - \frac{3\pi}{2} - \theta \leq \alpha_k - \frac{3\pi}{2} - \theta < \alpha_k - \alpha_q \leq (k - q)\Delta \leq N\Delta \leq \frac{\pi}{2} \]
and consequently,
\[ \cos(\theta_{\sigma(k)} - \theta) = \sin(\theta_{\sigma(k)} - 3\pi/2 - \theta) \leq \sin((k - q)\Delta). \]

Hence and by (3.17) we obtain
\[ \Re(\overline{u}v_{\sigma(k)}) \leq \sin((k - q)\Delta), \quad k \in \mathbb{Z}_{q+1,q+q,N}. \] (3.21)

If \( 1 \leq k \leq p - N \) or \( q + N + 1 \leq k \leq n \), then clearly \( \Re(\overline{u}v_{\sigma(k)}) \leq 1 \). Combining this with (3.19), (3.20) and (3.21) we see that
\[ \sum_{k=1}^{n} \Re(\overline{u}v_{\sigma(k)}) \leq \sum_{k=p+1-N}^{p} \sin((p + 1 - k)\Delta) + \sum_{k=q+1}^{q+N} \sin((k - q)\Delta) \] (3.22)
\[ + (p - N) + (n - q - N) \]
\[ = 2 \sum_{k=1}^{N} \sin(k\Delta) + n - 2N - (q - p). \]

Since \( \pi < \alpha_{q+1} - \alpha_{p-1} \leq (q - p + 2)\Delta \), we deduce from (3.15) that \( 2N \leq q - p + 1 \). Combining this with (3.22) we get
\[ \sum_{k=1}^{n} \Re(\overline{u}v_{\sigma(k)}) \leq 2 \sum_{k=1}^{N} \sin(k\Delta) + n - 2N - (2N - 1) \]
\[ = n + 1 - 4N + 2 \frac{\sin((N+1)\Delta/2)}{\sin(\pi/2)} \sin(\frac{N\Delta}{2}). \]
Hence and by (3.2),
\[ S \leq n + 1 - 4N + 2 \frac{\sin(\frac{(N+1)\Delta}{2}) \sin(\frac{N\Delta}{2})}{\sin(\frac{\Delta}{2})}. \]

Theorem 3.1 now shows that
\[ |F(z)| \leq 1 - \left( 4N - 1 - 2 \frac{\sin(\frac{(N+1)\Delta}{2}) \sin(\frac{N\Delta}{2})}{\sin(\frac{\Delta}{2})} \right) p(z), \quad F \in \mathcal{F}, \; z \in \mathbb{D}, \quad (3.23) \]
where \( N \) and \( p(z) \) are defined by (3.15) and (3.3), respectively. Applying now Corollary 3.3 we derive from (3.23) the following estimation of radial type
\[ |F(z)| \leq 1 - \left( 4N - 1 - 2 \frac{\sin(\frac{(N+1)\Delta}{2}) \sin(\frac{N\Delta}{2})}{\sin(\frac{\Delta}{2})} \right) P\left[ \chi_I, |\cdot| \right], \quad F \in \mathcal{F}, \; z \in \mathbb{D}, \quad (3.24) \]
where \( \delta \) is given by the formula (3.6).

References

The Schwarz type inequality for harmonic functions

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NIERÓWNOŚCI TYPU SCHWARZA DLA FUNKCJI HARMONICZNYCH W KOLE JEDNOSTKOWYM SPEŁNIAJĄCYCH PEWIEN WARUNEK SEKTOROWY

S t r e s z c z e n i e

Niech $T_1$, $T_2$ i $T_3$ będą łukami domkniętymi, zawartymi w okręgu jednostkowym $\mathbb{T}$, o tej samej długości $2\pi/3$ i pokrywającymi $\mathbb{T}$. W pracy [3] D. Partyka and J. Zając otrzymali dokładne oszacowanie modułu $|F(z)|$ dla $z \in \mathbb{D}$, gdzie $\mathbb{D}$ jest kołem jednostkowym, zaś $F$ jest funkcją harmoniczną o wartościach zespolonych koło $\mathbb{D}$ w siebie, spełniających następujący warunek sektorowy: dla każdego $k \in \{1, 2, 3\}$ i prawie każdego $z \in T_k$ granica radialna funkcji $F$ w punkcie $z$ należy do sektora kątowego będącego otoczką wypukłą zbioru $\{0\} \cup T_k$. W tym artykule rozważamy ogólniejszy przypadek, gdzie trzy łuki są zastąpione przez skończony układ łuków domkniętych $T_1, T_2, \ldots, T_n$ zawartych w $\mathbb{T}$, o dodatniej długości, całkowitej długości $2\pi$ i pokrywających $\mathbb{T}$.

Słowa kluczowe: całka Poissona, funkcje harmoniczne, lemat Schwarz, odwzorowania harmoniczne