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Dedicated to the memory of
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OPTION PRICING IN CRR MODEL WITH TIME DEPENDENT PARAMETERS FOR TWO PERIODS OF TIME - PART II

Summary

In the second part of the paper we prove the convergence of option prices in the presented model to the price that is given by some formula corresponding to the Black-Scholes formula.

Keywords and phrases: Cox-Ross-Rubinstein model (CRR model), Black-Scholes formula, option pricing

4. Proofs

In the proof of Lemma 3.1 we use a certain version of Central Limit Theorem that is an immediate consequence of the Lindeberg-Feller Theorem.

Theorem 4.1. *For $n = 1, 2, 3,$ fix $0 < p_n < 1, q_n = 1 - p_n$. Let for every $n \in \mathbb{N}, A_{n,1}, A_{n,2}, ..., A_{n,n}$ be such independent events that $P_n(A_{n,i}) = p_n$ and S_n be the random variable describing the number of events $A_{n,i}$ that occur, so $S_n = \sum_{i=1}^{n} \mathbb{I}_{A_{n,i}}, P_n(S_n = k) = \binom{n}{k} \cdot p_n^k \cdot q_n^{n-k}, E[S_n] = n \cdot p_n, D^2[S_n] = n \cdot p_n \cdot q_n$. Assume that $\lim_{n\to\infty} p_n = p \in (0,1)$ and $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. Then*

$$P_n\left(\frac{S_n - np_n}{\sqrt{n \cdot p_n \cdot q_n}} \leq x_n\right) \overset{n\to\infty}{\longrightarrow} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-0,5u^2} du := \phi(x).$$

Proof of Lemma 3.1. Let us notice that taking in (3.1.1):

$$\tilde{C}_{1,n}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}) = \mathbb{I}_{(-\infty, s'_{1,k}]}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l})$$

we get

$$C_{0,n}(s_0) = \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot \mathbb{I}_{(-\infty, s'_{1,k}]}\left(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}\right) = \frac{1}{e^{r_1}} P(S_n \le l'_0),$$

where S_n is the random variable having Bernoulli distribution with the probability $p_{1,n}^*$ of the success in each sample that describes the number of the upper stock price's changes during the first period of time (from the moment 0 to the moment 1); l'_0 is the solution of the equation $s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l} = s'_{1,k}$, so

$$l'_0 = \frac{\ln s'_{1,k} - \ln s_0 + \sigma_1 \sqrt{n}}{2 \frac{\sigma_1}{\sqrt{n}}}. \tag{1}$$

From Theorem 4.1 we have:

$$0 \le \lim_{n\to\infty} C_{0,n}(s_0) = \lim_{n\to\infty} \frac{1}{e^{r_1}} P(S_n \le l'_0) = \frac{1}{e^{r_1}} \phi\left(\lim_{n\to\infty} \frac{l'_0 - np_{1,n}^*}{\sqrt{np_{1,n}^* q_{1,n}^*}}\right). \tag{2}$$

By [6, lemmas 2.2.1 and 2.2.3] we know that for $n \to \infty$: $\sqrt{p_{1,n}^* q_{1,n}^*} \to \frac{1}{2}$ and $\sqrt{n}(1 - 2p_{1,n}^*) \to \frac{1}{2}\sigma_1 - \frac{r_1}{\sigma_1}$. Hence from (1) we obtain:
$\lim_{n\to\infty} \frac{l'_0 - np_{1,n}^*}{\sqrt{np_{1,n}^* q_{1,n}^*}} = \lim_{n\to\infty} \frac{\ln \frac{s'_{1,k}}{s_0} + \sigma_1\sqrt{n}(1-2p_{1,n}^*)}{2\sigma_1 \sqrt{p_{1,n}^* q_{1,n}^*}} = \frac{\ln \frac{s'_{1,k}}{s_0}}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1$. From (2) we get

$$\lim_{n\to\infty} C_{0,n}(s_0) = \frac{1}{e^{r_1}} \phi\left(\frac{\ln \frac{s'_{1,k}}{s_0}}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1\right). \tag{3}$$

Estimate (3.1.2) gives us:

$$\underline{\lim}_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot C_{1,n}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}) \ge$$
$$\ge \underline{\lim}_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \sum_{k=1}^{\infty} x_k \mathbb{I}_{(s'_{1,k}, s''_{1,k}]}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}).$$

Since all the terms are nonnegative, we can change the order of summation, so

$$\underline{\lim}_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \sum_{k=1}^{\infty} x_k \mathbb{I}_{(s'_{1,k}, s''_{1,k}]}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}) =$$
$$= \underline{\lim}_{n\to\infty} \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l})$$

and for every $M \in \mathbb{N}$ we get the following estimate:

$$\begin{aligned}
&\varliminf_{n\to\infty} \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{({s'}_{1,k},{s''}_{1,k}]}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) \geq \\
&\geq \varliminf_{n\to\infty} \frac{1}{e^{r_1}} \sum_{k=1}^{M} x_k \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{({s'}_{1,k},{s''}_{1,k}]}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) \geq \\
&\geq \frac{1}{e^{r_1}} \sum_{k=1}^{M} x_k \varliminf_{n\to\infty} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{({s'}_{1,k},{s''}_{1,k}]}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}).
\end{aligned}$$

Since M is arbitrary, we have

$$\begin{aligned}
&\varliminf_{n\to\infty} \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{({s'}_{1,k},{s''}_{1,k}]}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) \geq \\
&\geq \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \varliminf_{n\to\infty} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{({s'}_{1,k},{s''}_{1,k}]}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}).
\end{aligned}$$

From this and (3) we obtain:

$$\begin{aligned}
&\varliminf_{n\to\infty} \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{({s'}_{1,k},{s''}_{1,k}]}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) \geq \\
&\geq \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \left(\phi\left(\frac{\ln \frac{s''_{1,k}}{s_0} - r_1}{\sigma_1} + \frac{\sigma_1}{2} \right) - \phi\left(\frac{\ln \frac{s'_{1,k}}{s_0} - r_1}{\sigma_1} + \frac{\sigma_1}{2} \right) \right) = \\
&= \left(\begin{array}{l} t''_k = \frac{\ln s''_{1,k}}{\sigma_1} \\ t'_k = \frac{\ln s'_{1,k}}{\sigma_1} \end{array} \right) = \\
&= \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \left(\phi\left(t''_k - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) - \phi\left(t'_k - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) \right) = \\
&= \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} x_k \int_{t'_k}^{t''_k} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) dt = \\
&= \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \int_{\mathbb{R}} x_k f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) \mathbb{I}_{({t'}_k,{t''}_k]}(t) dt.
\end{aligned}$$

Finally, from the Lebesgue Monotone Convergence Theorem we have

$$\varliminf_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot C_{1,n}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) \geq$$

$$\geq \frac{1}{e^{r_1}} \int\limits_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1\right) \sum_{k=1}^{\infty} x_k \mathbb{I}_{(t'_k, t''_k]}(t)dt =$$

$$= \frac{1}{e^{r_1}} \int\limits_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \sum_{k=1}^{\infty} x_k \mathbb{I}_{(\frac{1}{\sigma_1}\ln s'_{1,k}, \frac{1}{\sigma_1}\ln s''_{1,k}]}(t)dt.$$

□

Proof of Lemma 3.2. Observe that intervals $(s'_{1,k}, s''_{1,k}], k = 0, 1, ...,$ are mutually disjoint and for every $v > 0$, the interval $(0, v]$ has common points with finite number M_v of these intervals. Fix the number $v > 0$ and let $(0, v] \subset \bigcup_{k=0}^{M_v} (s'_{1,k}, s''_{1,k}]$. Then

$$\sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot C_{1,n}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) = \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot$$

$$\cdot \left[C_{1,n}(s_0 u_{1,n}^{l} d_{1,n}^{n-l}) \mathbb{I}_{(0,v]}\left(s_0 u_{1,n}^{l} d_{1,n}^{n-l}\right) + C_{1,n}(s_0 u_{1,n}^{l} d_{1,n}^{n-l}) \mathbb{I}_{(v,\infty]}\left(s_0 \cdot^{u}_{1,n} \cdot d_{1,n}^{n-l}\right)\right] \overset{(3.2.1)}{\leq}$$

$$\leq \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \sum_{k=1}^{M_v} \tilde{x_k} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]}\left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) +$$

$$+ \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} C_{1,n}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) \mathbb{I}_{(v,\infty]}\left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right).$$

The proof is divided into two steps.
I. We will show that

$$\varlimsup_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \sum_{k=1}^{M_v} \tilde{x_k} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]}\left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) \leq$$

(1) $$\leq \frac{1}{e^{r_1}} \int\limits_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \sum_{k=1}^{\infty} \tilde{x_k} \mathbb{I}_{(\frac{1}{\sigma_1}\ln s'_{1,k}, \frac{1}{\sigma_1}\ln s''_{1,k}]}(t)dt.$$

Indeed. Since the sums have the finite number of summands, we can change the order of summation and moreover change the limit and the sum:

$$\overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \sum_{k=1}^{M_v} \tilde{x_k} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]} \left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) =$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{k=1}^{M_v} \tilde{x_k} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]} \left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) \leq$$

$$\leq \frac{1}{e^{r_1}} \sum_{k=1}^{M_v} \tilde{x_k} \overline{\lim_{n\to\infty}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]} \left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) \leq$$

$$\leq \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \tilde{x_k} \overline{\lim_{n\to\infty}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]} \left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right).$$

Taking into account the inequality above and the formula (3) in the proof of Lemma 3.1, we have

$$\overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{k=1}^{M_v} \tilde{x_k} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]} \left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) \leq$$

$$\leq \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \tilde{x_k} \overline{\lim_{n\to\infty}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \mathbb{I}_{(s'_{1,k}, s''_{1,k}]} \left(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}\right) =$$

$$= \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \tilde{x_k} \left(\phi\left(\frac{\ln \frac{s''_{1,k}}{s_0}}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) - \phi\left(\frac{\ln \frac{s'_{1,k}}{s_0}}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) \right) =$$

$$= \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \tilde{x_k} \left(\phi\left(t''_k - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) - \phi\left(t'_k - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{1}{2}\sigma_1 \right) \right) =$$

$$= \frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \tilde{x_k} \int_{t'_k}^{t''_k} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2} \right)(t)dt.$$

As in the proof of Lemma 3.1 using The Lebesgue Theorem we get:

$$\frac{1}{e^{r_1}} \sum_{k=1}^{\infty} \tilde{x_k} \int_{t'_k}^{t''_k} f\left(t - \frac{\ln s_0 + r_1}{\sigma_1} + \frac{\sigma_1}{2} \right) dt =$$

$$= \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0 + r_1}{\sigma_1} + \frac{\sigma_1}{2} \right) \sum_{k=1}^{\infty} \tilde{x_k} \mathbb{I}_{(t'_{1,k}, t''_{1,k}]}(t)dt =$$

$$= \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0 + r_1}{\sigma_1} + \frac{\sigma_1}{2} \right) \sum_{k=1}^{\infty} \tilde{x_k} \mathbb{I}_{(\frac{1}{\sigma_1} \ln s'_{1,k}, \frac{1}{\sigma_1} \ln s''_{1,k}]}(t)dt.$$

Thus, we have shown that for every $v > 0$, (1) holds.

II. We will show that $\forall \varepsilon > 0 \; \exists v_\varepsilon > 0 \; \forall v > v_\varepsilon$

$$\overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} C_{1,n}(s_0 u_{1,n}^l d_{1,n}^{n-l}) \mathbb{I}_{(v,\infty)} \left(s_0 u_{1,n}^l d_{1,n}^{n-l}\right) \le \varepsilon. \tag{2}$$

Indeed. With the notation

$$C_1(s_1) := s_1 \phi\left(\frac{\ln \frac{s_1}{K} + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{K}{e^{r_2}} \phi\left(\frac{\ln \frac{s_1}{K} + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right)$$

and using the uniform convergence of $C_{1,n}$ to C_1 ([3]) we have for every $\varepsilon > 0$ for n larger than some n_ε (where n_ε is independent of v):

$$\sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} C_{1,n}(s_0 u_{1,n}^l d_{1,n}^{n-l}) \mathbb{I}_{(v,\infty)} \left(s_0 u_{1,n}^l d_{1,n}^{n-l}\right) \le$$

$$\le \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} \left[C_1(s_0 u_{1,n}^l d_{1,n}^{n-l}) + 0,5\varepsilon\right] \mathbb{I}_{(v,\infty)} \left(s_0 u_{1,n}^l d_{1,n}^{n-l}\right).$$

Notice that $C_1(s_1) \le s_1$. Hence

$$\sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} C_{1,n}(s_0 u_{1,n}^l d_{1,n}^{n-l}) \mathbb{I}_{(v,\infty)} \left(s_0 u_{1,n}^l d_{1,n}^{n-l}\right) \le$$

$$\le \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} \left[s_0 u_{1,n}^l d_{1,n}^{n-l} + 0,5\varepsilon\right] \mathbb{I}_{(v,\infty)} \left(s_0 u_{1,n}^l d_{1,n}^{n-l}\right) =$$

$$= E\left[(S(1,n) + 0,5\varepsilon)\mathbb{I}_{(v,\infty)}\left(S(1,n)\right)\right] \le E\left[(S(1,n))\mathbb{I}_{(v,\infty)}\left(S(1,n)\right)\right] + 0,5\varepsilon, \tag{3}$$

where $S(1,n)$ signifies the stock price after one period of time (after n moments of the portfolios change), $S(1,0) = s_0$.

We will show that $\lim_{n\to\infty} E\left[(S(1,n))\mathbb{I}_{(v,\infty)}\left(S(1,n)\right)\right] = E\left[(s(1))\mathbb{I}_{(v,\infty)}\left(s(1)\right)\right]$, where $s(1) := s_0 e^{r_1 - 0,5\sigma_1^2 + \sigma_1 Z}$, Z is the random variable with normal distribution $N(0,1)$. From Corollary 5.1 we have: $S(1,n) \overset{n\to\infty}{\longrightarrow} s(1)$ in distribution. By Skorochod's Theorem (Theorem 5.1), there is a probability space $(\tilde{\Omega}, \tilde{F}, \tilde{P})$, a sequence of random variables $(\tilde{S}(1,n))_{n=1}^{\infty}$ and a random variable $\tilde{s}(1)$ such as $\forall n \in N$, $\tilde{S}(1,n) \overset{d}{=} S(1,n)$, $\tilde{s}(1) \overset{d}{=} s(1)$ and $\tilde{S}(1,n) \overset{a.s.}{\longrightarrow} \tilde{s}(1)$. Notice that for every x, $\tilde{P}(\tilde{s}(1) = x) = 0$. Then $h(\tilde{S}(1,n)) \overset{a.s.}{\longrightarrow} h(\tilde{s}(1))$, where $h : [0,\infty) \longrightarrow [0,\infty)$, $h(x) := x\mathbb{I}_{[0,v]}(x)$. Since $0 \le h(x) \le v$ for each $x \in [0,\infty)$, from the Lebesgue Dominated Convergence Theorem we conclude that $\tilde{E}[h(\tilde{S}(1,n))] \overset{n\longrightarrow\infty}{\longrightarrow} \tilde{E}[h(\tilde{s}(1))]$, where $\tilde{E}$ is the expected value by $\tilde{P}$. Thus

$$E[h(S(1,n))] = \tilde{E}[h(\tilde{S}(1,n))] \overset{n\longrightarrow\infty}{\longrightarrow} \tilde{E}[h(\tilde{s}(1))] = E[h(s(1))] \tag{4}$$

Let us observe that

$$E[S(1,n)] = E\left[S(1,n)\mathbb{I}_{[0,v]}\left(S(1,n)\right)\right] + E\left[S(1,n)\mathbb{I}_{(v,\infty)}\left(S(1,n)\right)\right],$$

$$E[s(1)] = E\left[s(1)\mathbb{I}_{[0,v]}\left(s(1)\right)\right] + E\left[s(1)\mathbb{I}_{(v,\infty)}\left(s(1)\right)\right].$$

From this, (4) and Collorary 5.1 we get

$$\lim_{n\to\infty} E\left[S(1,n)\mathbb{I}_{(v,\infty)}\left(S(1,n)\right)\right] = E\left[s(1)\mathbb{I}_{(v,\infty)}\left(s(1)\right)\right]. \tag{5}$$

Now we prove that

$$\lim_{v\to\infty} E\left[s(1)\mathbb{I}_{(v,\infty)}\left(s(1)\right)\right] = 0. \tag{6}$$

Indeed. Let $v_m \overset{m\longrightarrow\infty}{\longrightarrow} \infty$. Then the random variable $\zeta_m := s(1)\mathbb{I}_{(v_m,\infty)}(s(1)) \overset{m\longrightarrow\infty}{\longrightarrow} 0$ a.s. and $\forall\omega\in\Omega\ 0 \leq \zeta_m(\omega) \leq s(1)$, where $E[s(1)] < \infty$, so from the Lebesgue Dominated Convergence Theorem we obtain (6).
From (3) and (5) it follows that for each $\varepsilon > 0$ and for all $v > 0$:

$$\overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} C_{1,n}\left(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}\right) \mathbb{I}_{(v,\infty)}\left(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}\right) \leq \tag{7}$$

$$\leq \frac{1}{e^{r_1}} \lim_{n\to\infty}\left(E\left[S(1,n)\mathbb{I}_{(v,\infty)}\left(S(1,n)\right)\right] + 0,5\varepsilon\right) = \frac{1}{e^{r_1}}\left(E\left[s(1)\mathbb{I}_{(v,\infty)}\left(s(1)\right)\right] + 0,5\varepsilon\right).$$

Then (2) follows from (6) and (7).
Finally, according to calculations in both parts od this proof we get for fixed $\varepsilon > 0$ and large $v > 0$ ($v > v_\varepsilon > 0$)

$$\overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} q_{1,n}^{*n-l} C_{1,n}\left(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}\right) \leq$$

$$\overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \sum_{k=1}^{M_v} \tilde{x_k}\mathbb{I}_{(s^{`}_{1,k}, s^{``}_{1,k}]}\left(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}\right) +$$

$$+ \overline{\lim_{n\to\infty}} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} C_{1,n}(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}) \mathbb{I}_{(v,\infty)}\left(s_0 \cdot u_{1,n}^l \cdot d_{1,n}^{n-l}\right) \leq$$

$$\leq \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0 + r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \sum_{k=1}^{\infty} \tilde{x_k}\mathbb{I}_{(\frac{1}{\sigma_1}\ln s^{`}_{1,k}, \frac{1}{\sigma_1}\ln s^{``}_{1,k}]}(t)dt + \varepsilon.$$

Since ε is arbitrary, we obtain the conclusion of Lemma 3.2. □

Proof of Corollary 3.1. Since the function $(\hat{c}_1(\cdot) - \varepsilon)^+$ is continuous and non-negative, the set $\Phi_\varepsilon(\cdot)$ is non-empty. Let $\varphi(\cdot) : \mathfrak{R} \to [0,\infty)$ be a step function taking a finite number of values and bounding from below the function $(\hat{c}_1(\cdot) - \varepsilon)^+$, (for all $t \in \mathfrak{R}$ $\varphi(t) \leq (\hat{c}_1(\cdot) - \varepsilon)^+$). Denote $t := \frac{\ln s_1}{\sigma_1}$. From the uniform convergence of CRR model to Black-Scholes model ([3]) we have:

$$\forall\varepsilon > 0 \quad \exists n_0 \in N \quad \forall n \geq n_0 \quad \forall t \in \mathfrak{R} \quad |\hat{c}_1(t) - \hat{C}_{1,n}(t)| \leq \varepsilon,$$

where

$$\hat{C}_{1,n}(t) = \frac{1}{e^{r_2}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{2,n}^{*l} \cdot q_{2,n}^{*n-l} \cdot (e^{t\sigma_1} \cdot u_{2,n}^{l} \cdot d_{2,n}^{n-l} - K)^+ = C_{1,n}(s_1), s_1 = e^{t\sigma_1}.$$

Fix $\varepsilon > 0$. Hence for $n \geq n_0$ we have for every $t \in \mathfrak{R}$:

$$(\hat{c}_1(t) - \varepsilon)^+ \leq \hat{C}_{1,n}(t) \leq \hat{c}_1(t) + \varepsilon.$$

Then

$$\forall \varphi \in \Phi_\varepsilon \;\; \forall t \in \mathfrak{R} \;\;, \varphi(t) \leq \hat{C}_{1,n}(t).$$

Let us note that we also have

$$\forall \varphi \in \Phi_\varepsilon \;\; \forall s_1 = e^{\sigma_1 t} > 0, \;\; \tilde{\varphi}(e^{\sigma_1 t}) = \tilde{\varphi}(s_1) \leq C_{1,n}(s_1),$$

where $\tilde{\varphi}(s_1) \equiv \varphi(\frac{\ln s_1}{\sigma_1}) = \varphi(t)$. From Lemma 3.1 we get

$$\forall \varphi \in \Phi_\varepsilon \quad \varliminf_{n\to\infty} C_{0,n}(s_0) \geq \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \varphi(t) dt,$$

and finally

$$\varliminf_{n\to\infty} C_{0,n}(s_0) \geq \sup_{\varepsilon>0, \varphi\in\Phi_\varepsilon} \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \varphi(t) dt.$$

□

Proof of Corollary 3.2. This proof is analogous to the proof of Corollary 3.1.

Proof of Theorem 3.1. From Corollary 3.1 and Corollary 3.2 we have

$$\varliminf_{n\to\infty} C_{0,n}(s_0) \geq \sup_{\varepsilon>0, \varphi\in\Phi_\varepsilon} \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \varphi(t) dt, \tag{1}$$

$$\varlimsup_{n\to\infty} C_{0,n}(s_0) \leq \inf_{\varepsilon>0, \psi\in\Psi_\varepsilon} \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \psi(t) dt. \tag{2}$$

We will prove the equalities (3), (4) below:

$$\sup_{\varepsilon>0, \varphi\in\Phi_\varepsilon} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \varphi(t) dt =$$

$$= \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \hat{c}_1(t) dt, \tag{3}$$

$$\inf_{\varepsilon>0,\psi\in\Psi_\varepsilon}\int_{-\infty}^{\infty} f\left(t-\frac{\ln s_0}{\sigma_1}-\frac{r_1}{\sigma_1}+\frac{\sigma_1}{2}\right)\psi(t)dt=$$

$$=\int_{-\infty}^{\infty} f\left(t-\frac{\ln s_0}{\sigma_1}-\frac{r_1}{\sigma_1}+\frac{\sigma_1}{2}\right)\hat{c}_1(t)dt, \tag{4}$$

where $\hat{c}_1$ is the function given by the formula (3.2), $\Psi_\varepsilon(\cdot)$ is a set of step functions defined in Corollary 3.1, $\Phi_\varepsilon(\cdot)$ is a set of step functions defined in Corollary 3.2.

The proof is divided into two steps.
I. Proof of (3). Because each continuous function $f:\mathfrak{R}\to[0,\infty)$ can be pointwisely approximated from below in non-decreasing way by step functions having finite ranges, so for each fixed $n\in N$ there exists a non-decreasing sequence of non-negative step functions $(\tilde{\varphi}_{n,m})_{m=1}^{\infty}$ such that $\tilde{\varphi}_{n,m}\nearrow(\hat{c}_1-\frac{1}{n})^+$ as $m\longrightarrow\infty$. Fix $m\in N$ and let $(\tilde{\varphi}_{n,m})_{m=1}^{\infty}$ be the sequence having the following terms:

$$\varphi_{1,m}=\tilde{\varphi}_{1,m},$$

$$\varphi_{2,m}=\max(\varphi_{1,m},\tilde{\varphi}_{2,m})=\max(\tilde{\varphi}_{1,m},\tilde{\varphi}_{2,m}),$$

$$\varphi_{n,m}=\max(\varphi_{n-1,m},\tilde{\varphi}_{n,m})=\max(\tilde{\varphi}_{1,m},\tilde{\varphi}_{2,m},...,\tilde{\varphi}_{n,m}),n\in N.$$

Because for the fixed $n\in N$, $(\tilde{\varphi}_{n,m})_{m=1}^{\infty}$ is non-decreasing sequence of step functions, so for the fixed $n\in N$ $(\varphi_{n,m})_{n=1}^{\infty}$ is also non-decreasing sequence of step functions pointwisely approximating the function $(\hat{c}_1-\frac{1}{n})^+$. Thus

$$\begin{array}{ccccccc}\varphi_{1,1}\le\varphi_{1,2}\le...\le\varphi_{1,m}\le... \overset{m\longrightarrow\infty}{\longrightarrow}(\hat{c}_1-\frac{1}{1})^+,\\ \leq\quad\leq\quad\leq\\ \cdots\\ \leq\quad\leq\quad\leq\\ \varphi_{n,1}\le\varphi_{n,2}\le...\le\varphi_{n,m}\le... \overset{m\longrightarrow\infty}{\longrightarrow}(\hat{c}_1-\frac{1}{n})^+,n\in N.\end{array} \tag{5}$$

Let μ be the distribution with the density $\tilde{f}$, where

$$\tilde{f}(t)=f(t-a),a=\frac{\ln s_0}{\sigma_1}+\frac{r_1}{\sigma_1}-\frac{\sigma_1}{2},t\in\mathfrak{R}, \tag{6}$$

so $\mu=N(a,1)$. We know that for each $n\in N$, $\varphi_{n,m}\overset{m\to\infty}{\longrightarrow}(\hat{c}_1-\frac{1}{n})^+$ pointwisely in $\mathfrak{R}$, so for each $n\in N$, $\varphi_{n,m}\overset{\mu}{\longrightarrow}(\hat{c}_1-\frac{1}{n})^+$ as $m\to\infty$. The convergence in probability is metrizable, for example with metric ρ (see [9], exercise 8, p.112). Hence for n-th row in (5) we have

$$\exists m(n) \in N \quad \forall m \geq m(n) \quad \rho\left((\hat{c}_1 - \frac{1}{n})^+, \varphi_{n,m}\right) < \frac{1}{n}. \tag{7}$$

Choose a subsequence $(\varphi_{n,m_n})_{n=1}^{\infty}$ that satisfies (7). We take in $(n+1)$-th step $m_{n+1} = \max(m(n+1), m_n)$, where $m(n+1)$ is a number definited in (7) for $n+1$. Then $\varphi_{n+1,m_{n+1}} \geq \varphi_{n,m_{n+1}} \geq \varphi_{n,m_n}$ (because $(\varphi_{n,m})_{n,m=1}^{\infty}$ is a non-decreasing sequence in a column and in a row in (5)). From (7), because $m_{n+1} \geq m(n+1)$, we get

$$\rho\left(\left(\hat{c}_1 - \frac{1}{n+1}\right)^+, \varphi_{n+1,m_{n+1}}\right)\Big) < \frac{1}{n+1}. \tag{8}$$

Let us observe that

$$\forall \varepsilon > 0 \quad \rho\left((\hat{c}_1 - \varepsilon)^+, \hat{c}_1\right) \leq \varepsilon. \tag{9}$$

From (8) and (9) we deduce

$$\rho\left(\hat{c}_1, \varphi_{n+1,m_{n+1}}\right) < \frac{2}{n+1}. \tag{10}$$

By the above, we have a non-decreasing sequence of non-negative step functions $(\varphi_{n,m_n})_{n=1}^{\infty}$ convergent in the measure μ to the function $\hat{c}_1$. Let us choose a subsequence $(\varphi_{n_k,m_{n_k}})$ convergent μ almost everywhere to $\hat{c}_1$ (the Riesz Theorem). Note that $\varphi_{n_k,m_{n_k}} \in \Phi_\varepsilon$, where $\varepsilon = \frac{1}{n_k}$. Then

$$\sup_{\varepsilon>0,\varphi\in\Phi_\varepsilon} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)\varphi(t)dt \geq$$

$$\geq \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)\varphi_{n_k,m_{n_k}}(t)dt = \int_{\mathfrak{R}} \varphi_{n_k,m_{n_k}} d\mu.$$

From the Lebesgue Monotone Convergence Theorem we get

$$\sup_{\varepsilon>0,\varphi\in\Phi_\varepsilon} \int_{\mathfrak{R}} \varphi d\mu \geq \lim_{n_k\nearrow\infty} \int_{\mathfrak{R}} \varphi_{n_k,m_{n_k}} d\mu = \int_{\mathfrak{R}} \hat{c}_1 d\mu. \tag{11}$$

Moreover, $\forall \varepsilon > 0 \quad \forall \varphi \in \Phi_\varepsilon, \quad \varphi \leq \hat{c}_1$ and $\int_{\mathfrak{R}} \varphi d\mu \leq \int_{\mathfrak{R}} \hat{c}_1 d\mu$, so

$$\sup_{\varepsilon>0,\varphi\in\Phi_\varepsilon} \int_{\mathfrak{R}} \varphi d\mu \leq \int_{\mathfrak{R}} \hat{c}_1 d\mu. \tag{12}$$

Then we deduce (3) from (11) and (12).

II. Proof of (4). Because each continuous function $g : [0,\infty) \to [0,\infty)$ can be pointwisely approximated from above in non-increasing way by step functions having countable ranges, so for each fixed $n \in N$ there exists a non-increasing sequence of non-negative step functions $(\tilde{\psi}_{n,m})_{m=1}^{\infty}$ that are elements of the set $\Psi_\varepsilon(\cdot)$ such that

$\tilde{\psi}_{n,m} \searrow (\hat{c}_1 + \frac{1}{n})^+$ as $m \longrightarrow \infty$.
Analogously as in the first step of the proof we get the non-increasing sequence of non-negative step functions $(\psi_{n,m_n})_{n=1}^{\infty}$ convergent in the measure μ to the function $\hat{c}_1$. Let us choose a subsequence convergent μ almost everywhere to $\hat{c}_1$. Note that $\psi_{n_k,m_{n_k}} \in \Psi_\varepsilon$, where $\varepsilon = \frac{1}{n_k}$. Then

$$\inf_{\varepsilon>0,\psi\in\Psi_\varepsilon} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)\psi(t)dt \leq$$

$$\leq \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)\psi_{n_k,m_{n_k}}(t)dt = \int_{\mathfrak{R}} \psi_{n_k,m_{n_k}} d\mu. \tag{13}$$

We will prove that we can find a sequence $(\psi_{n,m})_{n,m=1}^{\infty}$ such that

$$\forall n,m \in N \quad \psi_{n,m} \leq \eta, \tag{14}$$

where $\int_{\mathfrak{R}} \eta d\mu < \infty$.
Let us notice that $0 < C_1(s_1) \leq s_1$ for $s_1 > 0$, then taking $C_1(0) = 0$ we obtain the continuous function on $[0, \infty)$. Hence for $s \in \left(\frac{k}{2^m}, \frac{k+1}{2^m}\right]$:

$$\bar{\psi}_{1,m}(s) \stackrel{df}{=} \max_{s_1\in\left(\frac{k}{2^m},\frac{k+1}{2^m}\right]} C_1(s_1) + 1 \leq \max_{s_1\in\left(\frac{k}{2^m},\frac{k+1}{2^m}\right]} s_1 + 1 \leq \frac{k+1}{2^m} + 1 \leq \frac{k}{2^m} + 2 \leq s + 2.$$

Hence $\bar{\psi}_{1,m}(s) \leq s + 2$ for each $s > 0$. What is more, the sequence $(\bar{\psi}_{1,m})_{m=1}^{\infty}$ is the non-increasing sequence (we take max for smaller intervals), so

$$\forall t \in \mathfrak{R} \quad \psi_{1,m}(t) \stackrel{df}{=} \bar{\psi}_{1,m}(e^{\sigma_1 t}) \leq e^{\sigma_1 t} + 2 \stackrel{df}{=} \eta(t).$$

Analogously we define for $n \in N$, $s \in (0, \infty)$:

$$\bar{\psi}_{n,m}(s) = \frac{1}{n} + (\bar{\psi}_{1,m}(s) - 1).$$

Thus for fixed $n \in N$, the sequence $(\bar{\psi}_{n,m})_{m=1}^{\infty}$ is non-increasing and

$$\lim_{m\to\infty} \bar{\psi}_{n,m}(s) = C_1(s) + \frac{1}{n}, \quad s > 0.$$

Moreover, for fixed $m \in N$, the sequence $(\bar{\psi}_{n,m})_{n=1}^{\infty}$ is also non-increasing.
Let

$$\forall m,n \in N \quad \psi_{n,m}(t) = \bar{\psi}_{n,m}(e^{\sigma_1 t}), t \in \mathfrak{R}.$$

Then $(\psi_{n,m})_{n=1}^{\infty}$ is non-increasing, so for each $t \in \mathfrak{R}$

$$\forall n \in N \quad \forall m \in N \quad \psi_{n,m}(t) \leq \psi_{1,m}(t) \leq \eta(t).$$

Moreover,

$$\int_{\mathfrak{R}} \eta d\mu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2 + e^{\sigma_1 t}) e^{-\frac{(t-a)^2}{2}} dt = 2 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(t-a-\sigma_1)^2}{2}} e^{a\sigma_1 + 0,5\sigma_1^2} dt =$$
$$= 2 + e^{a\sigma_1 + 0,5\sigma_1^2} = 2 + e^{r_1 + \ln s_0} = 2 + s_0 e^{r_1} < \infty.$$

The second to last equality follows from (6).
Thus we have proved (14).
From (13)-(14) and the Lebesguea Monotone Convergence Theorem we have

$$\inf_{\varepsilon>0, \psi\in\Psi_\varepsilon} \int_{\mathfrak{R}} \psi d\mu \le \lim_{n_k \uparrow \infty} \int_{\mathfrak{R}} \psi_{n_k, m_{n_k}} d\mu = \int_{\mathfrak{R}} \hat{c}_1 d\mu. \tag{15}$$

We also have

$$\forall \varepsilon > 0 \quad \forall \psi \in \Psi_\epsilon \quad \psi \ge \hat{c}_1 \quad \text{and} \quad \int_{\mathfrak{R}} \psi d\mu \ge \int_{\mathfrak{R}} \hat{c}_1 d\mu.$$

Thus

$$\inf_{\varepsilon>0, \psi\in\Psi_\varepsilon} \int_{\mathfrak{R}} \psi d\mu \ge \int_{\mathfrak{R}} \hat{c}_1 d\mu. \tag{16}$$

From (15) and (16) we get (4). □

In the proof of Theorem 2.1 we use the following lemma.

Lemma 4.1. *Let X and Y be independent continuous random variables, f_X be the probability density function of the random variable X, $\phi_Y(a) := \int_{-\infty}^{a} f_Y(u)du$ be the cumulative distribution function of the random variable Y. Then*

$$P(X + Y < a) = \int_{-\infty}^{\infty} f_X(s) \cdot \phi_Y(a - s) ds.$$

Proof of Theorem 2.1. From Theorem 3.1 we have

$$\lim_{n\to\infty} C_{0,n}(s_0) = \lim_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot C_{1,n}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}) =$$

$$= \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \cdot$$

$$\cdot \left[e^{\sigma_1 t} \phi\left(\frac{\sigma_1 t - \ln K + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{K}{e^{r_2}} \phi\left(\frac{\sigma_1 t - \ln K + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right)\right] dt =$$

$$= \frac{1}{e^{r_1}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)^2\right)} \cdot$$

$$\cdot \left[e^{\sigma_1 t} \phi\left(\frac{\sigma_1 t - \ln K + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{K}{e^{r_2}} \phi\left(\frac{\sigma_1 t - \ln K + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right)\right] dt =$$

$$= \frac{1}{e^{r_1}\sqrt{2\pi}} \int_{-\infty}^{\infty} s_0 \cdot e^{r_1} e^{-\frac{1}{2}\left(\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} - \frac{\sigma_1}{2}\right)^2\right)} \cdot \phi\left(\frac{\sigma_1 t - \ln K + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) dt -$$

$$- \frac{K}{\sqrt{2\pi} e^{r_1 + r_2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)^2\right)} \cdot \phi\left(\frac{\sigma_1 t - \ln K + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right) dt.$$

Substituting $w = \frac{\sigma_1 t}{\sigma_2}$ we obtain:

$$\lim_{n\to\infty} C_{0,n}(s_0) =$$

$$= \frac{s_0 \sigma_2 e^{r_1}}{\sqrt{2\pi}\sigma_1 e^{r_1}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(\frac{\sigma_2 w}{\sigma_1} - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} - \frac{\sigma_1}{2}\right)^2\right)} \cdot \phi\left(w + \frac{-\ln K + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) dw -$$

$$- \frac{K\sigma_2}{\sqrt{2\pi}\sigma_1 e^{r_1 + r_2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(\frac{\sigma_2 w}{\sigma_1} - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right)^2\right)} \cdot \phi\left(w + \frac{-\ln K + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right) dw =$$

$$= s_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}\left(\frac{w - (\ln s_0 + r_1 + 0,5\sigma_1^2)\frac{1}{\sigma_2}}{\frac{\sigma_1}{\sigma_2}}\right)^2} \left(1 - \phi\left(-w + \frac{\ln K - r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right)\right) dw -$$

$$\frac{K}{e^{r_1 + r_2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}\left(\frac{w - (\ln s_0 + r_1 - 0,5\sigma_1^2)\frac{1}{\sigma_2}}{\frac{\sigma_1}{\sigma_2}}\right)^2} \left(1 - \phi\left(-w + \frac{\ln K - r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right)\right) dw.$$

Let us notice that

$$f_X(w) := \frac{1}{\sqrt{2\pi}\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}\left(\frac{w-(\ln s_0 + r_1 + 0{,}5\sigma_1^2)\frac{1}{\sigma_2}}{\frac{\sigma_1}{\sigma_2}}\right)^2}$$

is a probability density function of the random variable X with the normal distribution $N\left(\left(\ln s_0 + r_1 + \frac{1}{2}\sigma_1^2\right)\frac{1}{\sigma_2}, \left(\frac{\sigma_1}{\sigma_2}\right)^2\right)$. If Y is the random variable with the normal distribution $N(0,1)$ independent of X, then from Lemma 4.1 we get

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}\left(\frac{w-(\ln s_0 + r_1 + 0{,}5\sigma_1^2)\frac{1}{\sigma_2}}{\frac{\sigma_1}{\sigma_2}}\right)^2} \phi\left(-w + \frac{\ln K - r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right) dw =$$
$$= P\left(X + Y < \frac{\ln K - r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right).$$

What is more, the random variable $\frac{X+Y-\left(\ln s_0 + r_1 + \frac{1}{2}\sigma_1^2\right)\frac{1}{\sigma_2}}{\sqrt{1+\left(\frac{\sigma_1}{\sigma_2}\right)^2}}$ has the normal distribution $N(0,1)$. Analogously

$$f_{\tilde{X}}(w) := \frac{1}{\sqrt{2\pi}\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}\left(\frac{w-(\ln s_0 + r_1 - 0{,}5\sigma_1^2)\frac{1}{\sigma_2}}{\frac{\sigma_1}{\sigma_2}}\right)^2}$$

is a probability density function of the random variable $\tilde{X}$ with the normal distribution $N\left(\left(\ln s_0 + r_1 - \frac{1}{2}\sigma_1^2\right)\frac{1}{\sigma_2}, \left(\frac{\sigma_1}{\sigma_2}\right)^2\right)$ and from Lemma 4.1 for the random variable Y with the normal distribution $N(0,1)$ independent of $\tilde{X}$:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}\left(\frac{w-(\ln s_0 + r_1 - 0{,}5\sigma_1^2)\frac{1}{\sigma_2}}{\frac{\sigma_1}{\sigma_2}}\right)^2} \phi\left(-w + \frac{\ln K - r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) dw =$$
$$= P\left(\tilde{X} + Y < \frac{\ln K - r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right),$$

and

$$\frac{\tilde{X} + Y - \left(\ln s_0 + r_1 - \frac{1}{2}\sigma_1^2\right)\frac{1}{\sigma_2}}{\sqrt{1+\left(\frac{\sigma_1}{\sigma_2}\right)^2}}$$

is the random variable with the normal distribution $N(0,1)$. Thus

$$\lim_{n\to\infty} C_{0,n}(s_0) = s_0\left(1 - P\left(X + Y < \frac{\ln K - r_2 - \frac{1}{2}\sigma_2^2}{\sigma_2}\right)\right) -$$
$$- \frac{K}{e^{r_1}e^{r_2}} \cdot \left(1 - P\left(\tilde{X} + Y < \frac{\ln K - r_2 + \frac{1}{2}\sigma_2^2}{\sigma_2}\right)\right) =$$
$$= s_0\left(1 - \phi\left(\frac{\ln K - r_2 - \frac{1}{2}\sigma_2^2 - \ln s_0 - r_1 - \frac{1}{2}\sigma_1^2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)\right) -$$
$$- \frac{K}{e^{r_1}e^{r_2}}\left(1 - \phi\left(\frac{\ln K - r_2 + \frac{1}{2}\sigma_2^2 - \ln s_0 - r_1 + \frac{1}{2}\sigma_1^2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)\right) =$$
$$= s_0 \cdot \phi(A \cdot \ln s_0 + B) - \frac{K}{e^{r_1} \cdot e^{r_2}} \cdot \phi(A \cdot \ln s_0 + \tilde{B}).$$

□

5. Appendix

A limit stock price in CRR model

Let us note that a stock price $S(n)$ after the first period of time (after n moments of the portfolio's change) is given by the formula: $S(n) = s_o^{\frac{\sigma_1}{\sqrt{n}}\sum_{j=1}^{n} V_{1,j}^{(n)}}$, where $V_{i,j}^{(n)} = \frac{\ln U_{i,j}^{(n)}}{\sigma_i/\sqrt{n}}, U_{i,j}^{(n)}$ is the random variable that takes the value $u_{i,n} = e^{\frac{\sigma_i}{\sqrt{n}}}$ with the probability $p_{i,n}^* = \frac{\hat{r}_{i,n} - d_{i,n}}{u_{i,n} - d_{i,n}} = \frac{e^{r_i/n} - e^{-\sigma_i/\sqrt{n}}}{e^{\sigma_i/\sqrt{n}} - e^{-\sigma_i/\sqrt{n}}}$ and the value $d_{i,n} = e^{\frac{-\sigma_i}{\sqrt{n}}}$ with the probability $q_{i,n}^* = 1 - p_{i,n}^* = \frac{u_{i,n} - \hat{r}_{i,n}}{u_{i,n} - d_{i,n}}$ ($i = 1$, because we consider only one period of time). Moreover, for fixed n, the random variables $(U_{i,j}^{(n)})_{j=1}^n$, $i, j = 1, 2$, are independent, so the random variables $(V_{i,j}^{(n)})_{j=1}^n$, $i, j = 1, 2$, are also independent. Thus

$$E\left[V_{i,j}^{(n)}\right] = E\left[\frac{\ln U_{i,j}^{(n)}}{\sigma_i/\sqrt{n}}\right] = \frac{\sqrt{n}}{\sigma_i}\left[\frac{\sigma_i}{\sqrt{n}}p_{i,n}^* - \frac{\sigma_i}{\sqrt{n}}q_{i,n}^*\right] = 2p_{i,n}^* - 1,$$

$$Var\left[V_{i,j}^{(n)}\right] = \frac{n}{\sigma_i^2}\left[\frac{\sigma_i^2}{n}p_{i,n}^* + \frac{\sigma_i^2}{n}q_{i,n}^*\right] - (2p_{i,n}^* - 1)^2 = 4p_{i,n}^* q_{i,n}^*.$$

We can easily check that random variables $(V_{i,j}^{(n)})_{j=1}^n$ satisfy Lapunov's Condtition, so from The Central Limit Theorem of Lapunov we get

$$\frac{\sum_{j=1}^{n} V_{i,j}^{(n)} - \sum_{j=1}^{n} EV_{i,j}^{(n)}}{\sqrt{4np_{i,n}^* q_{i,n}^*}} \xrightarrow[n\longrightarrow\infty]{d} Z \sim N(0,1).$$

Thus

$$\frac{\sigma_i\left(\sum\limits_{j=1}^{n} V_{i,j}^{(n)} - n(2p_{i,n}^* - 1)\right)}{\sqrt{4np_{i,n}^* q_{i,n}^*}} \underset{n\longrightarrow\infty}{\overset{d}{\longrightarrow}} \sigma_i Z \sim N(0,\sigma_i^2).$$

From Lemma 2.2.2 [6] we have $4p_{i,n}^* q_{i,n}^* \overset{n\longrightarrow\infty}{\longrightarrow} 1$, so

$$\frac{\sigma_i\sqrt{4p_{i,n}^* q_{i,n}^*}\left(\sum\limits_{j=1}^{n} V_{i,j}^{(n)} - n(2p_{i,n}^* - 1)\right)}{\sqrt{4np_{i,n}^* q_{i,n}^*}} \underset{n\longrightarrow\infty}{\overset{d}{\longrightarrow}} \sigma_i Z \sim N(0,\sigma_i^2).$$

From Lemma 2.2.3 [6] we know that $\sqrt{n}(2p_{i,n}^* - 1) \overset{n\longrightarrow\infty}{\longrightarrow} \frac{r}{\sigma_i} - \frac{1}{2}\sigma_i$. Consequently:

$$\frac{\sigma_i}{\sqrt{n}}\sum_{j=1}^{n} V_{i,j}^{(n)} \underset{n\longrightarrow\infty}{\overset{d}{\longrightarrow}} \sigma_i Z + r_i - \frac{1}{2}\sigma_i^2 \sim N(r_i - \frac{1}{2}\sigma_i^2, \sigma_i^2).$$

What is more,

$$\ln S(n) \underset{n\longrightarrow\infty}{\overset{d}{\longrightarrow}} \ln s_0 + r_1 - \frac{1}{2}\sigma_1^2 + \sigma_1 \cdot Z, \text{ where } Z \sim N(0,1),$$

and

$$S(n) \underset{n\longrightarrow\infty}{\overset{d}{\longrightarrow}} s_0 \cdot e^{r_1 - \frac{1}{2}\sigma_1^2 + \sigma_1 \cdot Z}.$$

Let $S(i,n)$ be a stock price after one period of time from the moment $i-1$ (after n moments of the portfolio's change), so $S(i,n)$ is a stock price at the moment i. Thus

$$S(i,n) = s_{i-1} \cdot e^{\frac{\sigma_i}{\sqrt{n}}\sum\limits_{j=1}^{n} V_{i,j}^{(n)}},$$

where s_{i-1} is the value of the positive random variable $S(i-1,n)$. We get

$$S(i,n) = S(i-1,n) \cdot e^{\frac{\sigma_i}{\sqrt{n}}\sum\limits_{j=1}^{n} V_{i,j}^{(n)}}.$$

By the above, we have the following lemma:

Lemma 5.1. *(Limit stock price) The limit distribution of a relative increment of a stock price after one period of time is a log-normal distribution:*

$$\frac{S(i,n)}{S(i-1,n)} \underset{n\longrightarrow\infty}{\overset{d}{\longrightarrow}} e^{r_i - \frac{1}{2}\sigma_i^2 + \sigma_i \cdot Z},$$

where $Z \sim N(0,1)$. Moreover,

$$E\left[\frac{S(i,n)}{S(i-1,n)}\right] = e^{r_i} = E\left[e^{r_i - \frac{1}{2}\sigma_i^2 + \sigma_i \cdot Z}\right].$$

Corollary 5.1. *For the first period of time (because s_0 in a non-negative constant) we have*

a) $\lim_{n\to\infty} S(1,n) = \lim_{n\to\infty} S(n) \overset{d}{=} s_0 \cdot e^{r_1 - \frac{1}{2}\sigma_1^2 + \sigma_1 \cdot Z} := s(1)$,

b) $E[S(1,n)] = E[S(n)] = s_0 \cdot e^{r_1} = E[s(1)]$.

Now, we recall Skorochod's Theorem.

Theorem 5.1. (Skorochod's Theorem, [1],[8]) *If* $\mu, \mu_1, \mu_2, \ldots$ *are probability distributions on* $(\mathfrak{R}, B(\mathfrak{R}))$ *and* $\mu_n \longrightarrow \mu$ *(weakly), then there exist random variables* $X, X_1, X_2, \ldots$ *with probability distributions* $\mu, \mu_1, \mu_2, \ldots$ *on the interval* $(0,1)$ *such that* $X_n(\omega) \longrightarrow X_n(\omega)$ *for each* $\omega \in (0,1)$.

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WYCENA OPCJI W MODELU CRR Z PARAMETRAMI ZALEŻNYMI OD CZASU DLA DWÓCH JEDNOSTEK CZASU - CZĘŚĆ II

S t r e s z c z e n i e

W drugiej części pracy udowodniono zbieżność ceny opcji w rozważanym modelu do ceny, która dana jest formułą podobną do wzoru Blacka-Scholesa.

Słowa kluczowe: model Coxa-Rossa-Rubinsteina, model CRR, wzór Blacka-Scholesa, wycena opcji