DOI: 10.26485/978-83-60655-92-4/5 ISBN 978-83-60655-92-4

(HYPER)COMPLEX SEMINAR 2021
IN MEMORIAM OF
PROF. JULIAN ŁAWRYNOWICZ

Tetiana M. Osipchuk

ON LINEAL CONVEXITY GENERALIZED TO CLIFFORD ALGEBRAS

Abstract

The notion of lineally convex domains in the finite-dimensional complex space and some of their properties are generalized to the finite-dimensional space $\mathcal{C}\ell_{p,q}^m$, $m \geq 2$, that is the Cartesian product of m universal Clifford algebras $\mathcal{C}\ell_{p,q}$ over the field of the real numbers. Namely, the separate necessary and sufficient conditions of the local $(\mathcal{C}\ell_{p,q}, d_1 d_2 \ldots d_m)$-lineal convexity of domains with smooth boundary are obtained for any collection $d_1 d_2 \ldots d_m$, where $d_j \in \{L, R\}$, $j = \overline{1, m}$. These conditions are a generalization of the well-known conditions of the local lineal convexity of a domain with smooth boundary, obtained by B. Zinoviev.

Keywords and phrases: Convex set, lineally convex set, $(\mathcal{C}\ell_{p,q}, d_1 d_2 \ldots d_m)$-lineally convex set, Clifford algebra, linear form, quadratic form, differential form, formal derivative.
Subject classification: 32F99, 52A30

1 Introduction

The notion of lineal convexity that is studied in the theory of functions of many complex variables was coined in 1935 by Heinrich Behnke and Ernst F. Peschl [1], but it has been actively used only since the 60s due to the works of André Martineau [2], [3] and Lev A. Aizenberg [4], [5] who considered the algebra of complex numbers $\mathbb{C}$ over the field of real numbers $\mathbb{R}$, and defined a lineally convex set in the finite-dimensional complex space $\mathbb{C}^n$, $n \geqslant 2$, independently in slightly different ways.

Consider a complex hyperplane

$$\Pi_{\mathbb{C}}(\boldsymbol{w}) := \left\{ \boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \boldsymbol{c}_j \, (\boldsymbol{z}_j - \boldsymbol{w}_j) = \boldsymbol{0}, (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n) \in \mathbb{C}^n \setminus \{\boldsymbol{0}\} \right\}.$$

Definition 1.1. (A. Martineau [2]) *A set $E \subset \mathbb{C}^n$ is said to be* ***lineally convex in the sense of Martineau*** *if its complement is a union of complex hyperplanes.*

The lineal convexity of a set $E \subset \mathbb{C}^n$ in the sense of Martineau is equivalent to the condition that, for any point $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2 \ldots, \boldsymbol{w}_n) \in \mathbb{C}^n \setminus E$, there exists a complex hyperplane $\Pi_{\mathbb{C}}(\boldsymbol{w})$ not intersecting E.

Definition 1.2. (L. Aizenberg [4]) *A domain $D \subset \mathbb{C}^n$ is said to be **lineally convex** if, for every boundary point $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2 \ldots, \boldsymbol{w}_n) \in \partial D$, there exists a complex hyperplane $\Pi_{\mathbb{C}}(\boldsymbol{w})$ not intersecting D.*

A domain lineally convex in the sense of Martineau is obviously lineally convex by Aizenberg. In [6] it is proved that there exist domains lineally convex by Aizenberg and not lineally convex in the sense of Martineau. The notion of lineal convexity in the sense of the Aizenberg definition is also known as ***weak lineal convexity*** [7], [8].

Definition 1.3. ([1, 9, 10]) *A domain $D \subset \mathbb{C}^n$ is said to be **locally lineally convex** if, for every boundary point $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2 \ldots, \boldsymbol{w}_n) \in \partial D$, there exists a complex hyperplane $\Pi_{\mathbb{C}}(\boldsymbol{w})$ passing through $\boldsymbol{w}$ but not intersecting D in some neighborhood of the point $\boldsymbol{w}$.*

There is also another definition of local lineal convexity:

Definition 1.4. ([11]) *An open set $D \subset \mathbb{C}^n$ is said to be **locally lineally convex in the sense of Kiselman** if, for every point $\boldsymbol{w} \in \mathbb{C}^n$, there exists a neighborhood U of $\boldsymbol{w}$ such that $D \cap U$ is lineally convex.*

Local lineal convexity in the sense of Kiselman implies local lineal convexity for all open sets. But there exists a bounded domain in $\mathbb{C}^2$ with Lipschitz boundary which is locally lineally convex but not locally lineally convex in the sense of Kiselman (see Example 4.4 in [11]).

H. Behnke and E. Peschl in [1] proved that global lineal convexity follows from the local one for bounded domains with a smooth boundary in $\mathbb{C}^2$. For the case of $\mathbb{C}^n$ this result was obtained in 1971 by Alexander P. Yuzhakov and Viachelsav P. Krivokolesko [9]. In the work [1], the separate necessary and sufficient analytical conditions of local lineal convexity of domains with smooth boundary in $\mathbb{C}^2$ were also obtained. In 1971 B. S. Zinoviev got a generalization of Behnke-Peschl conditions for the case $\mathbb{C}^n$, $n \geqslant 2$, in terms of nonnegativity and positivity of the differential of the second order of a real function defining a regular domain with smooth boundary, respectively. Moreover, the sign of the differential is determined on the boundary of the domain and on the vectors of a complex hyperplane tangent to the domain [10]. In 1998 Christer O. Kiselman managed to obtain the criterion of lineal convexity of a bounded domain in the space $\mathbb{C}^n$ with boundary of the class C^2 in terms of nonnegativity of the differential of the second order of the function defining the domain [8]. In 2008 Lars Hörmander improved Kiselman's result by loosening conditions imposed on the boundary of the domain [12].

In 1980s, the theory of lineally convex sets begins to be generalized to the spaces of hypercomplex numbers by Henzel A. Mkrtchyan and Yuri B. Zelinskii [13], [14].

Conditions similar to those of Zinoviev were obtained for the algebra of real quaternions [15], the algebra of real generalized quaternions [16], and Clifford algebras [17]. Moreover, all these papers consider hyperplanes with equations, where constants are multiplied by the variables either only on the right or only on the left.

The present paper considers the space $\mathcal{C}\ell^m_{p,q}$, $m \geq 2$, that is the Cartesian product of m universal Clifford algebras $\mathcal{C}\ell_{p,q}$ over the field of the real numbers. The main purpose of this paper is to obtain analytical conditions similar to those of Zinoviev on the vectors of the hyperplanes in the space $\mathcal{C}\ell^m_{p,q}$ with all possible equations, where in some terms the constants are multiplied by the variables on the right and in the remaining terms on the left. In chapter 2 the real linear and quadratic forms are presented in terms of the elements of Clifford algebra $\mathcal{C}\ell_{p,q}$ and a generalization of the complex formal partial derivatives to the algebra $\mathcal{C}\ell_{p,q}$ is obtained. In chapter 3 the notion of lineal convexity and the conditions of local lineal convexity are generalized to the space $\mathcal{C}\ell^m_{p,q}$.

2 Real linear and quadratic forms in Clifford algebras

Consider the universal Clifford algebra $\mathcal{C}\ell_{p,q}$, $p, q \in \mathbb{Z}$, $p, q \geq 0$, $p + q = n > 0$ [18], which is associative over the field of the real numbers, with the identity, and generated by the elements $\{s_j\}_{j=1}^n$ satisfying the conditions

$$s_j^2 = \begin{cases} 1, & j = 1, 2, \dots, p, \\ -1, & j = p+1, \dots, p+q; \end{cases}$$
$$s_j s_k + s_k s_j = 0, \qquad j \neq k. \tag{1}$$

The basis of Clifford algebra is constructed as follows. For every $\alpha := \{\alpha_1, \alpha_2 \dots, \alpha_k\} \subset N$, where $N := \{1, \dots, n\}$ and

$$1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq n,$$

we define

$$e_\emptyset := 1,\ e_\alpha := s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k},\ e_N := s_1 s_2 \dots s_n.$$

Then the set of all elements $\{e_\alpha : \alpha \subset N\}$ is the basis of Clifford algebra $\mathcal{C}\ell_{p,q}$ and $\dim \mathcal{C}\ell_{p,q} = 2^n$. Consider some properties of the basis elements. It is easy to see that

$$e_\alpha^2 = \pm 1, \quad \alpha \subset N.$$

Indeed, $e_\emptyset^2 = 1$. For the other $\alpha \subset N$, using formulas (1), we obtain:

$$e_\alpha^2 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k} s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k} = (-1)^{\frac{1}{2}k(k-1)} s_{\alpha_1} s_{\alpha_1} s_{\alpha_2} s_{\alpha_2} \dots s_{\alpha_k} s_{\alpha_k} =$$
$$= (-1)^{\frac{1}{2}k(k-1)+b},$$

where b is the number of multipliers s_{α_p}, $\alpha_p \in \alpha$, of the product $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ such that $s_{\alpha_p}^2 = -1$. Thus, each element e_α has the inverse element

$$e_\alpha^{-1} = \frac{e_\alpha}{e_\alpha^2} = \pm e_\alpha. \tag{2}$$

Let $\sharp\alpha$ be the number of the elements of the set α. Then, considering (1),

$$s_j e_\alpha = (-1)^{\sharp\alpha} e_\alpha s_j, \quad j \notin \alpha, \tag{3}$$

$$s_j e_\alpha = (-1)^{\sharp\alpha - 1} e_\alpha s_j, \quad j \in \alpha, \tag{4}$$

for any s_j, $j = \overline{1, n}$, and any $\alpha \subset N$.

Let $\sharp\alpha\beta$ be the number of the elements of the set $\alpha \cap \beta$ for any $\alpha, \beta \subset N$. Then, considering conditions (3), (4), we obtain:

$$e_\alpha e_\beta = (-1)^{\sharp\alpha(\sharp\beta - \sharp\alpha\beta)} \cdot (-1)^{(\sharp\alpha - 1)\sharp\alpha\beta} e_\beta e_\alpha = (-1)^{(\sharp\alpha\sharp\beta - \sharp\alpha\beta)} e_\beta e_\alpha. \tag{5}$$

For the convenience, we numerate the basis elements of Clifford algebra from 0 to $2^n - 1$ and represent each element $\boldsymbol{a} \in \mathcal{C}\ell_{p,q}$ as:

$$\boldsymbol{a} = \sum_{k=0}^{2^n - 1} a_k \boldsymbol{e}_k, \tag{6}$$

where $a_k \in \mathbb{R}$ and $\boldsymbol{e}_k$, $k = \overline{0, 2^n - 1}$, are the elements of the basis, moreover, $\boldsymbol{e}_0 = 1$.

Consider the vector space

$$\mathcal{C}\ell_{p,q}^m := \underbrace{\mathcal{C}\ell_{p,q} \times \mathcal{C}\ell_{p,q} \times \ldots \times \mathcal{C}\ell_{p,q}}_{m}$$

with the elements $\boldsymbol{z} := (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_m) \in \mathcal{C}\ell_{p,q}^m$, where

$$\boldsymbol{z}_j := \sum_{k=0}^{2^n - 1} x_k^j \boldsymbol{e}_k \in \mathcal{C}\ell_{p,q}, \quad x_k^j \in \mathbb{R}, \quad k = \overline{0, 2^n - 1}, \quad j = \overline{1, m}.$$

Let

$$\|\boldsymbol{z}\| = \sqrt{\sum_{j=1}^{m} \sum_{k=0}^{2^n - 1} \left| x_k^j \right|^2}$$

and $U(\boldsymbol{w}) = \{\boldsymbol{z} : \|\boldsymbol{z} - \boldsymbol{w}\| < \delta\}$.

Consider the following $2^n \times 2^n$ matrices defined recursively:

$$\Gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \Gamma_2 = \begin{pmatrix} \Gamma_1 & \Gamma_1 \\ \Gamma_1 & -\Gamma_1 \end{pmatrix}, \ \ldots$$

$$\ldots, \Gamma_n = \begin{pmatrix} \Gamma_{n-1} & \Gamma_{n-1} \\ \Gamma_{n-1} & -\Gamma_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & .. & 1 & 1 \\ 1 & -1 & .. & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & .. & (-1)^{n-1} & (-1)^{n-1} \\ 1 & -1 & .. & (-1)^{n-1} & (-1)^n \end{pmatrix}. \tag{7}$$

Prove by the induction that

$$\Gamma_n^{-1} = \frac{1}{2^n}\Gamma_n. \tag{8}$$

$$\Gamma_1\Gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the following property of the block matrices (see [19]):

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}, \tag{9}$$

where A_{ij}, B_{ij}, $i,j = 1,2$, are square matrices of the same order. Let E_n be the $2^n \times 2^n$ unit matrix and Θ_n be the $2^n \times 2^n$ matrix consisting of zeros. Then

$$\Gamma_2\Gamma_2 = \begin{pmatrix} \Gamma_1 & \Gamma_1 \\ \Gamma_1 & -\Gamma_1 \end{pmatrix}\begin{pmatrix} \Gamma_1 & \Gamma_1 \\ \Gamma_1 & -\Gamma_1 \end{pmatrix} = \begin{pmatrix} 2\Gamma_1\Gamma_1 & \Theta_1 \\ \Theta_1 & 2\Gamma_1\Gamma_1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2^2\,E_1 & \Theta_1 \\ \Theta_1 & 2^2\,E_1 \end{pmatrix} = 2^2 E_2.$$

Suppose $\Gamma_{n-1}\Gamma_{n-1} = 2^{n-1}E_{n-1}$. Then, considering (9),

$$\Gamma_n\Gamma_n = \begin{pmatrix} \Gamma_{n-1} & \Gamma_{n-1} \\ \Gamma_{n-1} & -\Gamma_{n-1} \end{pmatrix}\begin{pmatrix} \Gamma_{n-1} & \Gamma_{n-1} \\ \Gamma_{n-1} & -\Gamma_{n-1} \end{pmatrix} =$$

$$= \begin{pmatrix} 2\Gamma_{n-1}\Gamma_{n-1} & \Theta_{n-1} \\ \Theta_{n-1} & 2\Gamma_{n-1}\Gamma_{n-1} \end{pmatrix} = \begin{pmatrix} 2^n\,E_{n-1} & \Theta_{n-1} \\ \Theta_{n-1} & 2^n\,E_{n-1} \end{pmatrix} = 2^n E_n.$$

This immediately implies the formula (8).

Now consider the following matrices

$$\boldsymbol{Z}_j = \begin{pmatrix} \boldsymbol{z}_j^{\boldsymbol{0}} \\ \boldsymbol{z}_j^{\boldsymbol{1}} \\ \dots \\ \boldsymbol{z}_j^{\boldsymbol{2^n-1}} \end{pmatrix}, \quad \boldsymbol{X}^j = \begin{pmatrix} x_0^j\boldsymbol{e}_0 \\ x_1^j\boldsymbol{e}_1 \\ \dots \\ x_{2^n-1}^j\boldsymbol{e}_{2^n-1} \end{pmatrix}, \quad j = \overline{1,m}.$$

Let

$$\boldsymbol{Z}_j = \Gamma_n\boldsymbol{X}^j. \tag{10}$$

Then

$$\boldsymbol{z}_j^{\boldsymbol{l}} := \gamma_{l0}x_0^j\boldsymbol{e}_0 + \gamma_{l1}x_1^j\boldsymbol{e}_1 + \dots + \gamma_{l(2^n-1)}x_{2^n-1}^j\boldsymbol{e}_{2^n-1},\ l = \overline{0,2^n-1},\ j = \overline{1,m},$$

where γ_{lp}, $l,p = \overline{0,2^n-1}$, are the elements of Γ_n.

From now on, for any element $\boldsymbol{a} \in \mathcal{C}\ell_{p,q}$, the elements $\boldsymbol{a}^{\boldsymbol{l}} \in \mathcal{C}\ell_{p,q}$, $l = \overline{0,2^n-1}$, with upper index l in bold are obtained from $\boldsymbol{a}$ by multiplying the elements of the lth row of the matrix Γ_n by the respective summands $a_k\boldsymbol{e}_k$ in the basis decomposition (6) of $\boldsymbol{a}$.

We obtain from (10) and (8):

$$\boldsymbol{X}^j = \frac{1}{2^n}\Gamma_n \boldsymbol{Z}_j.$$

That is to say,

$$x_l^j = \frac{1}{2^n}\boldsymbol{e}_l^{-1}\sum_{p=0}^{2^n-1}\gamma_{lp}\boldsymbol{z}_j^{\boldsymbol{p}} = \frac{1}{2^n}\left(\sum_{p=0}^{2^n-1}\gamma_{lp}\boldsymbol{z}_j^{\boldsymbol{p}}\right)\boldsymbol{e}_l^{-1},\ j=\overline{1,m},\ l=\overline{0,2^n-1}. \tag{11}$$

Consider a real linear form

$$\sum_{j=1}^{m}\sum_{l=0}^{2^n-1}a_l^j x_l^j,$$

where $a_l^j \in \mathbb{R}$, $a_l^j = \text{const}$, $j=\overline{1,m}$, $l=\overline{0,2^n-1}$. Substitute x_l^j for their expressions from (11) and group together the respective components with $\boldsymbol{z}_j^{\boldsymbol{p}}$, $j=\overline{1,m}$, $l=\overline{0,2^n-1}$ fixing j and p. Then we obtain

$$\sum_{l=0}^{2^n-1}a_l^j x_l^j = \frac{1}{2^n}\sum_{l=0}^{2^n-1}a_l^j\boldsymbol{e}_l^{-1}\sum_{p=0}^{2^n-1}\gamma_{lp}\boldsymbol{z}_j^{\boldsymbol{p}} = \sum_{p=0}^{2^n-1}\left(\frac{1}{2^n}\sum_{l=0}^{2^n-1}\gamma_{lp}a_l^j\boldsymbol{e}_l^{-1}\right)\boldsymbol{z}_j^{\boldsymbol{p}} =$$

$$= \sum_{p=0}^{2^n-1}\boldsymbol{a}_j^p\boldsymbol{z}_j^{\boldsymbol{p}},\ j=\overline{1,m},$$

or

$$\sum_{l=0}^{2^n-1}a_l^j x_l^j = \frac{1}{2^n}\sum_{l=0}^{2^n-1}a_l^j\left(\sum_{p=0}^{2^n-1}\gamma_{lp}\boldsymbol{z}_j^{\boldsymbol{p}}\right)\boldsymbol{e}_l^{-1} = \sum_{p=0}^{2^n-1}\boldsymbol{z}_j^{\boldsymbol{p}}\frac{1}{2^n}\sum_{l=0}^{2^n-1}\gamma_{lp}a_l^j\boldsymbol{e}_l^{-1} =$$

$$= \sum_{p=0}^{2^n-1}\boldsymbol{z}_j^{\boldsymbol{p}}\boldsymbol{a}_j^p,\ j=\overline{1,m},$$

where

$$\boldsymbol{a}_j^p = \frac{1}{2^n}\sum_{l=0}^{2^n-1}\gamma_{lp}a_l^j\boldsymbol{e}_l^{-1},\ j=\overline{1,m},\ p=\overline{0,2^n-1}. \tag{12}$$

Then

$$\sum_{j=1}^{m}\sum_{l=0}^{2^n-1}a_l^j x_l^j = \sum_{j=1}^{m}A_j,$$

where

$$A_j = \sum_{p=0}^{2^n-1}\boldsymbol{z}_j^{\boldsymbol{p}}\boldsymbol{a}_j^p = \sum_{p=0}^{2^n-1}\boldsymbol{a}_j^p\boldsymbol{z}_j^{\boldsymbol{p}},\ j=\overline{1,m}.$$

Let us rewrite the expression of $\boldsymbol{a}_j^p$ in terms of indices i, q.

$$\boldsymbol{a}_i^q = \frac{1}{2^n}\sum_{k=0}^{2^n-1}\gamma_{kq}a_k^i\boldsymbol{e}_k^{-1},\ i=\overline{1,m},\ q=\overline{0,2^n-1}.$$

Now we consider a real quadratic form

$$\sum_{j,i=1}^{m}\sum_{l,k=0}^{2^n-1}a_{lk}^{ji}x_l^jx_k^i, \tag{13}$$

where $a_{lk}^{ji}\in\mathbb{R}$ are the elements of a symmetric $2^nm\times 2^nm$ matrix

$$\left(a_{lk}^{ji}\right),\quad a_{lk}^{ji}=a_{kl}^{ij},\ j,i=\overline{1,m},\ k,l=\overline{0,2^n-1}, \tag{14}$$

presented as follows:

$$\left(a_{lk}^{ji}\right)=\begin{pmatrix} a^{11} & a^{12} & .. & a^{1m}\\ a^{21} & a^{22} & .. & a^{2m}\\ \vdots & \vdots & \ddots & \vdots\\ a^{m1} & a^{m2} & .. & a^{mm}\end{pmatrix},$$

where

$$a^{ji}=\begin{pmatrix} a_{00}^{ji} & a_{01}^{ji} & .. & a_{0(2^n-1)}^{ji}\\ a_{10}^{ji} & a_{11}^{ji} & .. & a_{1(2^n-1)}^{ji}\\ \vdots & \vdots & \ddots & \vdots\\ a_{(2^n-1)0}^{ji} & a_{(2^n-1)1}^{ji} & .. & a_{(2^n-1)(2^n-1)}^{ji}\end{pmatrix},\ \ i,j=\overline{1,m}.$$

Multiplying $\boldsymbol{a}_j^p$ by $\boldsymbol{a}_i^q$ and replacing the products $a_l^ja_k^i$ with the elements a_{lk}^{ji} of the matrix (14) we get the following elements of the Clifford algebra:

$$\boldsymbol{a}_{ji}^{pq}=\frac{1}{2^{2n}}\sum_{l,k=0}^{2^n-1}\gamma_{lp}\gamma_{kq}a_{lk}^{ji}\boldsymbol{e}_l^{-1}\boldsymbol{e}_k^{-1},\ j,i=\overline{1,m},\ p,q=\overline{0,2^n-1}. \tag{15}$$

Then

$$\begin{aligned}\sum_{l,k=0}^{2^n-1}a_{lk}^{ji}x_l^jx_k^i &= \sum_{j,i=1}^{m}\sum_{l,k=0}^{2^n-1}a_{lk}^{ji}\left(\boldsymbol{e}_l^{-1}\frac{1}{2^n}\sum_{p=0}^{2^n-1}\gamma_{lp}\boldsymbol{z}_j^{\boldsymbol{p}}\right)x_k^i = \\ =\sum_{l,k=0}^{2^n-1}a_{lk}^{ji}\boldsymbol{e}_l^{-1}\frac{1}{2^n}\sum_{p=0}^{2^n-1}\gamma_{lp}x_k^i\boldsymbol{z}_j^{\boldsymbol{p}} &= \sum_{l,k=0}^{2^n-1}a_{lk}^{ji}\boldsymbol{e}_l^{-1}\frac{1}{2^n}\sum_{p=0}^{2^n-1}\gamma_{lp}\left(\boldsymbol{e}_k^{-1}\frac{1}{2^n}\sum_{q=0}^{2^n-1}\gamma_{kq}\boldsymbol{z}_i^{\boldsymbol{q}}\right)\boldsymbol{z}_j^{\boldsymbol{p}}\\ &=\frac{1}{2^{2n}}\sum_{p,q=0}^{2^n-1}\sum_{l,k=0}^{2^n-1}\gamma_{lp}\gamma_{kq}a_{lk}^{ji}\boldsymbol{e}_l^{-1}\boldsymbol{e}_k^{-1}\boldsymbol{z}_i^{\boldsymbol{q}}\boldsymbol{z}_j^{\boldsymbol{p}}=\sum_{p,q=0}^{2^n-1}\boldsymbol{a}_{ji}^{pq}\boldsymbol{z}_i^{\boldsymbol{q}}\boldsymbol{z}_j^{\boldsymbol{p}},\quad i,j=\overline{1,m}.\end{aligned}$$

Similary, substituting in turn x_l^j, x_k^i in (13) for their different expressions from (11), it can be obtained that

$$\sum_{l,k=0}^{2^n-1} a_{lk}^{ji} x_l^j x_k^i = \sum_{p,q=0}^{2^n-1} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_i^{\boldsymbol{q}} = \sum_{p,q=0}^{2^n-1} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{a}_{ji}^{pq}, \quad i,j=\overline{1,m}.$$

Thus, the real quadratic form (13) can be expressed in terms of the elements $\boldsymbol{z}_j^{\boldsymbol{p}}$, $\boldsymbol{z}_i^{\boldsymbol{q}}$, $\boldsymbol{a}_{ji}^{pq}$ as follows:

$$\sum_{j,i=1}^{m} A_{ji}, \quad \text{where} \quad A_{ji} = \sum_{p,q=0}^{2^n-1} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{z}_j^{\boldsymbol{p}} = \sum_{p,q=0}^{2^n-1} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_i^{\boldsymbol{q}} = \sum_{p,q=0}^{2^n-1} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{a}_{ji}^{pq}.$$

Moreover,

$$\sum_{j,i=1}^{m} \sum_{p,q=0}^{2^n-1} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{z}_i^{\boldsymbol{q}} = \sum_{j,i=1}^{m} \sum_{p,q=0}^{2^n-1} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_j^{\boldsymbol{p}} = \sum_{j,i=1}^{m} \sum_{p,q=0}^{2^n-1} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{a}_{ji}^{pq} \neq \sum_{j,i=1}^{m} \sum_{l,k=0}^{2^n-1} a_{lk}^{ji} x_l^j x_k^i,$$

Indeed,

$$\sum_{p,q=0}^{2^n-1} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{z}_i^{\boldsymbol{q}} = \frac{1}{2^{2n}} \sum_{p,q=0}^{2^n-1} \sum_{l,k=0}^{2^n-1} \gamma_{lp}\gamma_{kq} a_{lk}^{ji} \boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1} \sum_{g=0}^{2^n-1} \gamma_{pg} x_g^j \boldsymbol{e}_g \sum_{h=0}^{2^n-1} \gamma_{qh} x_h^i \boldsymbol{e}_h =$$
$$= \frac{1}{2^{2n}} \sum_{l,k=0}^{2^n-1} \sum_{g,h=0}^{2^n-1} a_{lk}^{ji} x_g^j x_h^i \boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1} \boldsymbol{e}_g \boldsymbol{e}_h \sum_{p,q=0}^{2^n-1} \gamma_{lp}\gamma_{kq}\gamma_{pg}\gamma_{qh}.$$

Considering the fact that

$$\frac{1}{2^n} \sum_{p=0}^{2^n-1} \gamma_{lp}\gamma_{pg} = \begin{cases} 1, & g=l, \\ 0, & g\neq l, \end{cases} \qquad \frac{1}{2^n} \sum_{q=0}^{2^n-1} \gamma_{kq}\gamma_{qh} = \begin{cases} 1, & h=k, \\ 0, & h\neq k, \end{cases}$$

by (8), we obtain

$$\sum_{p,q=0}^{2^n-1} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{z}_i^{\boldsymbol{q}} = \sum_{l,k=0}^{2^n-1} a_{lk}^{ji} x_l^j x_k^i \boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1} \boldsymbol{e}_l \boldsymbol{e}_k.$$

Thus,

$$\sum_{p,q=0}^{2^n-1} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{z}_i^{\boldsymbol{q}} \neq \sum_{l,k=0}^{2^n-1} a_{lk}^{ji} x_l^j x_k^i,$$

since there exist indices l,k such that $\boldsymbol{e}_l \boldsymbol{e}_k \neq \boldsymbol{e}_k \boldsymbol{e}_l$. Moreover, considering (2), (5), $\boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1} = \pm \boldsymbol{e}_k^{-1} \boldsymbol{e}_l^{-1}$.

Similarly,

$$\sum_{p,q=0}^{2^n-1} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{a}_{ji}^{pq} \boldsymbol{z}_j^{\boldsymbol{p}} = \sum_{l,k=0}^{2^n-1} a_{lk}^{ji} x_l^j x_k^i \boldsymbol{e}_k \boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1} \boldsymbol{e}_l,$$

$$\sum_{p,q=0}^{2^n-1} \boldsymbol{z}_j^{\boldsymbol{p}} \boldsymbol{z}_i^{\boldsymbol{q}} \boldsymbol{a}_{ji}^{pq} = \sum_{l,k=0}^{2^n-1} a_{lk}^{ji} x_l^j x_k^i \boldsymbol{e}_l \boldsymbol{e}_k \boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1}.$$

Let $\rho(\boldsymbol{z}) = \rho(z) : \mathbb{R}^{m2^n} \to \mathbb{R}$, $\boldsymbol{z} \in \mathcal{C}\ell_{p,q}^m$, $z \in \mathbb{R}^{m2^n}$, have the continuous partial derivatives of the first and the second order at a point $w \in \mathbb{R}^{m2^n}$. Then the function ρ is twice continuously differentiable at the point w and its full differentials of the first and the second order are defined as follows:

$$d\rho(w) = \sum_{j=1}^{m} \sum_{l=0}^{2^n-1} \frac{\partial \rho(w)}{\partial x_l^j} dx_l^j, \qquad d^2\rho(w) = \sum_{j,i=1}^{m} \sum_{l,k=0}^{2^n-1} \frac{\partial^2 \rho(w)}{\partial x_k^i \partial x_l^j} dx_l^j \, dx_k^i.$$

Present $d\rho(w)$, $d^2\rho(w)$ in terms of the elements of $\mathcal{C}\ell_{p,q}$. Let

$$d\boldsymbol{z}_j^{\boldsymbol{p}} := \sum_{l=0}^{2^n-1} \gamma_{pl} dx_l^j \boldsymbol{e}_l, \quad j = \overline{1,m}, \ p = \overline{0, 2^n-1}.$$

Let $a_l^j = \dfrac{\partial \rho(w)}{\partial x_l^j}$, $\boldsymbol{a}_j^p = \dfrac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}}}$ in (12) and $a_{lk}^{ji} = \dfrac{\partial^2 \rho(w)}{\partial x_l^j \partial x_k^i}$, $\boldsymbol{a}_{ji}^{pq} = \dfrac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}} \partial \boldsymbol{z}_i^{\boldsymbol{q}}}$ in (15), $\boldsymbol{w} \in \mathcal{C}\ell_{p,q}^m$, $p, q = \overline{0, 2^n-1}$. Then

$$\frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}}} := \frac{1}{2^n} \sum_{l=0}^{2^n-1} \gamma_{lp} \frac{\partial \rho(w)}{\partial x_l^j} \boldsymbol{e}_l^{-1}, \ j = \overline{1,m}, \ p = \overline{0, 2^n-1}, \tag{16}$$

$$\frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}} \partial \boldsymbol{z}_i^{\boldsymbol{q}}} := \frac{1}{2^{2n}} \sum_{l,k=0}^{2^n-1} \gamma_{lp} \gamma_{kq} \frac{\partial^2 \rho(w)}{\partial x_l^j \partial x_k^i} \boldsymbol{e}_l^{-1} \boldsymbol{e}_k^{-1}, \ j, i = \overline{1,m}, \ p, q = \overline{0, 2^n-1}.$$

And

$$d\rho(\boldsymbol{w}) = \sum_{j=1}^{m} D_j, \text{ where } D_j = \sum_{p=0}^{2^n-1} \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}}} d\boldsymbol{z}_j^{\boldsymbol{p}} = \sum_{p=0}^{2^n-1} d\boldsymbol{z}_j^{\boldsymbol{p}} \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}}}, \tag{17}$$

$$d^2\rho(\boldsymbol{w}) = \sum_{j,i=1}^{m} D_{ji}, \tag{18}$$

where

$$D_{ji} = \sum_{q,p=0}^{2^n-1} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}} \partial \boldsymbol{z}_i^{\boldsymbol{q}}} d\boldsymbol{z}_i^{\boldsymbol{q}} \, d\boldsymbol{z}_j^{\boldsymbol{p}} = \sum_{q,p=0}^{2^n-1} d\boldsymbol{z}_j^{\boldsymbol{p}} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}} \partial \boldsymbol{z}_i^{\boldsymbol{q}}} d\boldsymbol{z}_i^{\boldsymbol{q}} = \sum_{q,p=0}^{2^n-1} d\boldsymbol{z}_i^{\boldsymbol{q}} \, d\boldsymbol{z}_j^{\boldsymbol{p}} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}} \partial \boldsymbol{z}_i^{\boldsymbol{q}}}.$$

We may also consider the function $\rho(z)$ as a real function of $m2^n$ variables of $\mathcal{C}\ell_{p,q}$. Indeed, substitut x_l^j, $j = \overline{1,m}$, $l = \overline{0, 2^n-1}$, for their values (11) in the expression of the function $\rho(z) = \rho(x_0^1, x_1^1, \ldots, x_{2^n-1}^m)$, then

$$\rho(z) = \rho(x_0^1(\boldsymbol{z}_1^{\boldsymbol{0}}, \boldsymbol{z}_1^{\boldsymbol{1}} \ldots, \boldsymbol{z}_1^{\boldsymbol{2^n-1}}), x_1^1(\boldsymbol{z}_1^{\boldsymbol{0}}, \boldsymbol{z}_1^{\boldsymbol{1}} \ldots, \boldsymbol{z}_1^{\boldsymbol{2^n-1}}), \ldots, \\ x_{2^n-1}^m(\boldsymbol{z}_m^{\boldsymbol{0}}, \boldsymbol{z}_m^{\boldsymbol{1}} \ldots, \boldsymbol{z}_m^{\boldsymbol{2^n-1}})) = \rho(\boldsymbol{z}, \boldsymbol{z}^{\boldsymbol{1}}, \ldots, \boldsymbol{z}^{\boldsymbol{2^n-1}}),$$

where $\boldsymbol{z}^{\boldsymbol{l}} = (\boldsymbol{z}_1^{\boldsymbol{l}}, \boldsymbol{z}_1^{\boldsymbol{l}}, \boldsymbol{z}_2^{\boldsymbol{l}} \ldots \boldsymbol{z}_m^{\boldsymbol{l}})$, $l = \overline{1, 2^n-1}$.

3 Generalized lineal convexity

Let $w = \left(w_0^1, w_1^1, \ldots, w_{2^n-1}^m\right) \in \mathbb{R}^{m2^n}$ and $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m) \in \mathcal{C}\ell_{p,q}^m$, where $\boldsymbol{w}_j = \sum\limits_{l=0}^{2^n-1} w_l^j \boldsymbol{e}_l$, $j = \overline{1,m}$. And let $\boldsymbol{s}_j = \sum\limits_{l=0}^{2^n-1} s_l^j \boldsymbol{e}_l = \sum\limits_{l=0}^{2^n-1} (x_l^j - w_l^j)\boldsymbol{e}_l = \boldsymbol{z}_j - \boldsymbol{w}_j$, $j = \overline{1,m}$.

For a collection $d_1 d_2 \ldots d_m$, where $d_j \in \{R, L\}$, $j = \overline{1,m}$, consider a hyperplane

$$\Pi_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \ldots d_m}(\boldsymbol{w}) := \left\{ \boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_m) \in \mathcal{C}\ell_{p,q}^m : \sum_{j=1}^m \boldsymbol{Q}_{d_j}^j = \boldsymbol{0}, \right.$$

$$\left. \boldsymbol{Q}_R^j = \boldsymbol{s}_j \boldsymbol{c}_j,\ \boldsymbol{Q}_L^j = \boldsymbol{c}_j \boldsymbol{s}_j,\ (\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m) \in \mathcal{C}\ell_{p,q}^m \setminus \{\boldsymbol{0}\} \right\}, \quad (19)$$

which is called $d_1 d_2 \ldots d_m$***-analytic***.

We say that two collections $d_1' d_2' \ldots d_m'$ and $d_1'' d_2'' \ldots d_m''$, where $d_j', d_j'' \in \{R, L\}$, $j = \overline{1,m}$, are different, if there exists at least one index k such that $d_k' \neq d_k''$. It is not difficult to prove that the number of all collections $d_1 d_2 \ldots d_m$ equals 2^m. Thus, for a fixed point $\boldsymbol{w} \in \mathcal{C}\ell_{p,q}^m$ and a fixed constant $(\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m) \in \mathcal{C}\ell_{p,q}^m \setminus \{\boldsymbol{0}\}$, there exist 2^m different hyperplanes $\Pi_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \ldots d_m}(\boldsymbol{w})$ in general case.

A $d_1 d_2 \ldots d_m$-analytic hyperplane $\Pi_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \ldots d_m}(\boldsymbol{w})$ (19) is called ***(locally) supporting*** for a domain $\Omega \subset \mathcal{C}\ell_{p,q}^m$ at a point $\boldsymbol{w} \in \partial\Omega$ if it does not intersect Ω (in some neighborhood of the point $\boldsymbol{w}$).

Definition 3.1. *A domain $\Omega \subset \mathcal{C}\ell_{p,q}^m$ is said to be* ***(locally)*** $(\mathcal{C}\ell_{p,q}, d_1 d_2 \ldots d_m)$***-lineally convex*** *if it has a (locally) supporting, $d_1 d_2 \ldots d_m$-analytic hyperplane $\Pi_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \ldots d_m}(\boldsymbol{w})$ at every point $\boldsymbol{w} \in \partial\Omega$.*

It is obvious that the notion of $(\mathbb{C}, d_1 d_2 \ldots d_m)$-lineal convexity is equivalent to the notion of lineal convexity in Definition 1.2 for any collection $d_1 d_2 \ldots d_m$.

We say that a hyperplane $\Pi_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \ldots d_m}(\boldsymbol{w})$ ***lies in a real hyperplane***

$$\Pi_{\mathbb{R}^{m2^n}}(w) := \left\{ \left(s_0^1, s_1^1, \ldots, s_{(2^n-1)}^n\right) \in \mathbb{R}^{m2^n} : \sum_{j=1}^m \sum_{l=0}^{2^n-1} a_l^j s_l^j = 0, \right.$$

$$\left. \left(a_0^1, a_1^1, \ldots, a_{(2^n-1)}^n\right) \in \mathbb{R}^{m2^n} \setminus \{0\} \right\} \quad (20)$$

if any vector $\boldsymbol{s}$ satisfying the equation of the hyperplane (19) satisfies the equation of the hyperplane (20).

Lemma 3.2. *For any real hyperplane $\Pi_{\mathbb{R}^{m2^n}}(w)$ and any collection $d_1 d_2 \ldots d_m$, the hyperplane $\Pi_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \ldots d_m}(\boldsymbol{w})$ (19) with coefficients $\boldsymbol{c}_j = \boldsymbol{a}_j^0$ (12) lies in $\Pi_{\mathbb{R}^{m2^n}}(w)$.*

Proof. Note that $\gamma_{l0} = 1$, $l = \overline{0, 2^n-1}$ (see (7)). Substitute the constants $\boldsymbol{c}_j$ in (19) for the values of $\boldsymbol{a}_j^0$ (12) and, after multiplying by $\boldsymbol{s}_j = \sum\limits_{p=0}^{2^n-1} s_p^j \boldsymbol{e}_p$, group together the terms with each basis element $\boldsymbol{e}_k$, $k = \overline{0, 2^n - 1}$, separately. Set the grouped expressions to zero. We obtain that the equation in (19) is equivalent to the system of 2^n real equations defining real hyperplanes in the $m2^n$-dimensional real space. Moreover, the equation obtained after grouping terms with the real unit $\boldsymbol{e}_0$ defines the real hyperplane $\Pi_{\mathbb{R}^{m2^n}}(w)$. The lemma is proved. □

Lemma 3.3. *Any convex domain in $\mathbb{R}^{m2^n}$ is $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-lineally convex in $\mathcal{C}\ell^m_{p,q}$ for any collection $d_1 d_2 \dots d_m$.*

Proof. Since a domain is convex, through its any boundary point w, there passes a real hyperplane $\Pi_{\mathbb{R}^{m2^n}}(w)$ (20) not intersecting the domain (see [20]). Then, for any collection $d_1 d_2 \dots d_m$, the hyperplane $\Pi^{d_1 d_2 \dots d_m}_{\mathcal{C}\ell_{p,q}}(\boldsymbol{w})$ (19), where $\boldsymbol{c}_j = \boldsymbol{a}^0_j$, $j = \overline{1, m}$ (12), does not intersect the domain by Lemma 3.2. The lemma is proved. □

The converse statement is not always true, which shows the following example. Consider a domain $D = D_1 \times \mathcal{C}\ell^{m-1}_{p,q} \subset \mathcal{C}\ell^m_{p,q}$, where $D_1 \subset \mathcal{C}\ell^1_{p,q}$ is a non-convex domain. The domain D is obviously non-convex. But it is $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-lineally convex for any collection $d_1 d_2 \dots d_m$, since any boundary point $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m) \in \partial D$ is such that $\boldsymbol{w}_1 \in \partial D_1$, $\boldsymbol{w}_k \in \mathcal{C}\ell_{p,q}$, $k = \overline{2, m}$, and a hyperplane with equation $\boldsymbol{s}_1 = \boldsymbol{z}_1 - \boldsymbol{w}_1 = \boldsymbol{0}$ is $d_1 d_2 \dots d_m$-analytic for any collection $d_1 d_2 \dots d_m$ and supporting for D at $\boldsymbol{w}$.

Now consider a domain

$$\Omega = \{\boldsymbol{z} \in \mathcal{C}\ell^m_{p,q} : \rho(\boldsymbol{z}) = \rho(\boldsymbol{z}, \boldsymbol{z}^{\boldsymbol{1}}, \dots, \boldsymbol{z}^{\boldsymbol{2^n-1}}) < 0\} \tag{21}$$

with the boundary $\partial\Omega = \{\boldsymbol{z} \in \mathcal{C}\ell^m_{p,q} : \rho(\boldsymbol{z}) = 0\}$, where the function $\rho : \mathcal{C}\ell^m_{p,q} \to \mathbb{R}$ is twice continuously differentiable in a neighborhood of $\partial\Omega$ with respect to its real variables and such that $\mathrm{grad}\rho \neq 0$ everywhere on $\partial\Omega$. Such a domain is called ***regular***.

A $d_1 d_2 \dots d_m$-analytic hyperplane $\Pi^{d_1 d_2 \dots d_m}_{\mathcal{C}\ell_{p,q}}(\boldsymbol{w})$, $\boldsymbol{w} \in \partial\Omega$, lying in the real hyperplane

$$T_{\mathbb{R}^{m2^n}}(w) := \left\{ \left(s^1_0, s^1_1, \dots, s^n_{(2^n-1)}\right) \in \mathbb{R}^{m2^n} : \sum_{j=1}^{m} \sum_{l=0}^{2^n-1} \frac{\partial \rho(w)}{\partial x^j_l} s^j_l = 0 \right\} \tag{22}$$

is called ***tangent*** to Ω at the point $\boldsymbol{w}$. Then, by Lemma 3.2 and considering (16), where $\dfrac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}^{\boldsymbol{0}}_j} = \dfrac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j}$, a $d_1 d_2 \dots d_m$-analytic hyperplane

$$T^{d_1 d_2 \dots d_m}_{\mathcal{C}\ell_{p,q}}(\boldsymbol{w}) := \left\{ (\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_m) \in \mathcal{C}\ell^m_{p,q} : \right.$$

$$\left. \sum_{j=1}^{m} \boldsymbol{Q}^j_{d_j} = \boldsymbol{0},\ \boldsymbol{Q}^j_R = \boldsymbol{s}_j \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j},\ \boldsymbol{Q}^j_L = \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j} \boldsymbol{s}_j \right\}$$

is tangent for any collection $d_1 d_2 \dots d_m$. If a regular domain $\Omega \subset \mathcal{C}\ell_{p,q}^m$ is (locally) $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-lineally convex for a fixed collection $d_1 d_2 \dots d_m$, then the tangent hyperplane $T_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \dots d_m}(\boldsymbol{w})$ is the unique $d_1 d_2 \dots d_m$-analytic hyperplane (locally) supporting for Ω at any boundary point $\boldsymbol{w}$ by smoothness of $\partial\Omega$ and considering the fact that

$$T_{\mathbb{R}^{m2^n}}(w) \equiv \Bigg\{ (\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_m) \in \mathcal{C}\ell_{p,q}^m : \\ \sum_{j=1}^{m} D_j = 0,\ D_j = \sum_{p=0}^{2^n-1} \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}}}\, \boldsymbol{s}_j^{\boldsymbol{p}} = \sum_{p=0}^{2^n-1} d\boldsymbol{s}_j^{\boldsymbol{p}}\, \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{p}}} \Bigg\}$$

by (17).

Theorem 3.4. *If a regular domain $\Omega \subset \mathcal{C}\ell_{p,q}^m$ is locally $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-lineally convex for a fixed collection $d_1 d_2 \dots d_m$, where $d_j \in \{R, L\}$, $j = \overline{1,m}$, then, for $\boldsymbol{w}$ and any vector $\boldsymbol{s} \in T_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \dots d_m}(\boldsymbol{w})$, $\|\boldsymbol{s}\| = 1$, the following inequality is true*

$$\sum_{i,j=1}^{m} D_{ij} \geqslant 0, \tag{23}$$

where

$$D_{ij} = \sum_{k,l=0}^{2^n-1} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}}\, \boldsymbol{s}_i^{\boldsymbol{k}} \boldsymbol{s}_j^{\boldsymbol{l}} = \sum_{k,l=0}^{2^n-1} \boldsymbol{s}_j^{\boldsymbol{l}} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}}\, \boldsymbol{s}_i^{\boldsymbol{k}} = \sum_{k,l=0}^{2^n-1} \boldsymbol{s}_i^{\boldsymbol{k}} \boldsymbol{s}_j^{\boldsymbol{l}} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}}.$$

If, for any point $\boldsymbol{w} \in \partial\Omega$ and any vector $\boldsymbol{s} \in T_{\mathcal{C}\ell_{p,q}}^{d_1 d_2 \dots d_m}(\boldsymbol{w})$, $\|\boldsymbol{s}\| = 1$,

$$\sum_{i,j=1}^{m} D_{ij} > 0, \tag{24}$$

then a regular domain $\Omega \subset \mathcal{C}\ell_{p,q}^m$ is locally $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-lineally convex.

Proof. The main idea of the proof of the theorem is similar to the one in the proof of Zinoviev Theorem.

Sufficiency. Consider the function $\rho(\boldsymbol{z})$ in (21) as the real function of $m2^n$ real variables and write its Taylor series in the neighborhood $U(\boldsymbol{w})$ of every point $\boldsymbol{w} \in \partial\Omega$:

$$\rho(\boldsymbol{z}) = \rho(\boldsymbol{w}) + \sum_{j=1}^{m} \sum_{l=0}^{2^n-1} \frac{\partial \rho(w)}{\partial x_l^j}\,(x_l^j - w_l^j) + \\ + \frac{1}{2} \sum_{i,j=1}^{m} \sum_{k,l=0}^{2^n-1} \frac{\partial^2 \rho(w)}{\partial x_k^i \partial x_l^j}\,(x_k^i - w_k^i)(x_l^j - w_l^j) + o(\|\boldsymbol{z} - \boldsymbol{w}\|^2), \qquad \boldsymbol{z} \to \boldsymbol{w}. \tag{25}$$

Notice that $\rho(\boldsymbol{w}) = 0$ at any boundary point $\boldsymbol{w}$. Since $\boldsymbol{s} \in T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\boldsymbol{w})$, therefore $\boldsymbol{s} \in T_{\mathbb{R}^{m2^n}}(w)$ (22) and the second summand in (25) also vanishes. Then, considering condition (18), we obtain:

$$\rho(\boldsymbol{z}) = \frac{1}{2}\left(\sum_{i,j=1}^{m} D_{ij}\right)\|\boldsymbol{z}-\boldsymbol{w}\|^2 + o(\|\boldsymbol{z}-\boldsymbol{w}\|^2),\ \boldsymbol{z}\to\boldsymbol{w}, \tag{26}$$

where

$$D_{ij} = \sum_{k,l=0}^{2^n-1} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}} \frac{(\boldsymbol{z}_i^{\boldsymbol{k}} - \boldsymbol{w}_i^{\boldsymbol{k}})(\boldsymbol{z}_j^{\boldsymbol{l}} - \boldsymbol{w}_j^{\boldsymbol{l}})}{\|\boldsymbol{z}-\boldsymbol{w}\|^2} =$$
$$= \sum_{k,l=0}^{2^n-1} \frac{(\boldsymbol{z}_i^{\boldsymbol{k}} - \boldsymbol{w}_i^{\boldsymbol{k}})}{\|\boldsymbol{z}-\boldsymbol{w}\|} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_i^{\boldsymbol{k}} \partial \boldsymbol{z}_j^{\boldsymbol{l}}} \frac{(\boldsymbol{z}_j^{\boldsymbol{l}} - \boldsymbol{w}_j^{\boldsymbol{l}})}{\|\boldsymbol{z}-\boldsymbol{w}\|} = \sum_{k,l=0}^{2^n-1} \frac{(\boldsymbol{z}_i^{\boldsymbol{k}} - \boldsymbol{w}_i^{\boldsymbol{k}})(\boldsymbol{z}_j^{\boldsymbol{l}} - \boldsymbol{w}_j^{\boldsymbol{l}})}{\|\boldsymbol{z}-\boldsymbol{w}\|^2} \frac{\partial^2 \rho(\boldsymbol{w})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}},$$

for any point $\boldsymbol{z} \in U(\boldsymbol{w}) \cap T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\boldsymbol{w})$.

Thus, $\rho(\boldsymbol{z}) \geqslant 0$ for any point $\boldsymbol{z} \in U(\boldsymbol{w}) \cap T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\boldsymbol{w})$ and any point $\boldsymbol{w} \in \partial\Omega$ by (24) and (26), which means local $(\mathcal{C}\ell_{p,q}, d_1 d_2 \ldots d_m)$-lineal convexity of the domain Ω.

Necessity. Let a regular domain Ω be locally $(\mathcal{C}\ell_{p,q}, d_1 d_2 \ldots d_m)$-lineally convex and for a point $\widetilde{\boldsymbol{w}} = (\widetilde{\boldsymbol{w}}_1, \widetilde{\boldsymbol{w}}_2, \ldots, \widetilde{\boldsymbol{w}}_n) \in \partial\Omega$ and for a vector $\boldsymbol{t} = (\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n) \in T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\widetilde{\boldsymbol{w}})$ the following inequality is true

$$\sum_{i,j=1}^{m} D_{ij} < 0, \tag{27}$$

where

$$D_{ij} = \sum_{k,l=0}^{2^n-1} \frac{\partial^2 \rho(\widetilde{\boldsymbol{w}})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}} \boldsymbol{t}_i^{\boldsymbol{k}} \boldsymbol{t}_j^{\boldsymbol{l}} = \sum_{k,l=0}^{2^n-1} \boldsymbol{t}_j^{\boldsymbol{l}} \frac{\partial^2 \rho(\widetilde{\boldsymbol{w}})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}} \boldsymbol{t}_i^{\boldsymbol{k}} = \sum_{k,l=0}^{2^n-1} \boldsymbol{t}_i^{\boldsymbol{k}} \boldsymbol{t}_j^{\boldsymbol{l}} \frac{\partial^2 \rho(\widetilde{\boldsymbol{w}})}{\partial \boldsymbol{z}_j^{\boldsymbol{l}} \partial \boldsymbol{z}_i^{\boldsymbol{k}}}.$$

On the other hand, for the points $\boldsymbol{z} \in U(\widetilde{\boldsymbol{w}}) \cap T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\widetilde{\boldsymbol{w}})$ the expansion (26) is valid. Thus, for the points $\widetilde{\boldsymbol{z}} = (\widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2, \ldots, \widetilde{\boldsymbol{z}}_n) \in U(\widetilde{\boldsymbol{w}}) \cap T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\widetilde{\boldsymbol{w}})$ corresponding to the tangent vector $\boldsymbol{t}$, where correspondence is defined by the relation $\boldsymbol{t}_i = (\widetilde{\boldsymbol{z}}_i - \widetilde{\boldsymbol{w}}_i)/\|\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{w}}\|$, $i = \overline{1,m}$, the inequality $\rho(\widetilde{\boldsymbol{z}}) < 0$ is true by (27), which contradicts the fact that the hyperplane $T^{d_1 d_2 \ldots d_m}_{\mathcal{C}\ell_{p,q}}(\widetilde{\boldsymbol{w}})$ is locally supporting for Ω at $\widetilde{\boldsymbol{w}}$. $\square$

References

[1] H. Behnke, E. Peschl, *Zur Theorie der Funktionen mehrerer komplexer Veränderlichen Konvexität in bezug auf analytische Ebenen im kleinen und großen* , Math. Ann., **111**:2, 158–177 (1935) (in German). https://doi.org/10.1007/BF01472211

[2] A. Martineau, *Sur la topologie des espaces de fonctions holomorphes* Math. Ann., **163**:1, 62–88 (1966) (in French). https://doi.org/10.1007/BF02052485

[3] A. Martineau, *Sur les équations aux dérivées partielles à coefficients constants avec second membre, dans le champ complexe.* C. R. Acad. Sci. Paris Sér, **163**:1, 62–88 (1967) (in French).

[4] L. A. Aizenberg, *Linear convexity in $\mathbb{C}^n$ and distribution of the singularities of holomorphic functions*, Bulletin of the Polish Academy of Sciences, Series Math., astr. and phys. sciences **15**:7, 487–495 (1967) (in Russian).

[5] L. A. Aizenberg, *Decomposition of holomorphic functions of several complex variables into partial fractions*, Sib Math J, **8**, 859–872 (1967). https://doi.org/10.1007/BF01040660

[6] L. A. Aizenberg, A. P. Yuzhakov, L. Ya. Makarova *On linear convexity in $\mathbb{C}^n$*, Sibirsk. Mat. Zh, **9**, no. 4, 731–746 (1968) (in Russian). http://mi.mathnet.ru/smj5559

[7] Ch. O. Kiselman, *Lineally convex Hartogs domains*, Acta Math. Vietnamica, **21**, 69–94 (1996).

[8] Ch. O. Kiselman, *A differential inequality characterizing weak lineal convexity*, Math. Ann. **311**:1, 1–10 (1998).

[9] A. P. Yuzhakov, V. P. Krivokolesko, *Some properties of lineally convex domains with smooth boundaries in $\mathbb{C}^n$*, Sib Math J, **12**, 323–327 (1971). https://doi.org/10.1007/BF00969055

[10] B. S. Zinoviev, *Analytic conditions and some questions of approximation of linear convex domains with smooth boundaries in the space $\mathbb{C}^n$*, Izv. vuzov, Math., no. 6, 61–69 (1971) (in Russian). http://mi.mathnet.ru/ivm3882

[11] Ch. O. Kiselman, *Weak lineal convexity*, Constructive Approximation of Functions. Banach Center Publications, Polish Academy of Sciences. **107**, 159–174 (2016). https://doi.org/10.4064/bc107-0-11

[12] L. Hörmander *Weak linear convexity and a related notion of concavity*, Math. Scand, **102**, 73–100 (2008). https://doi.org/10.7146/math.scand.a-15052

[13] H. A. Mkrtchyan, *Hypercomplex-convex domains with smooth boundary*, Doc. AN USSR, Ser. A :3, 15–17 (1986) (in Russian).

[14] Yu. B. Zelinskii, H. A. Mkrtchyan, *On extremal points and hypercomplex-convex domains*, Doc. AN USSR, **311**:6, 1299–1302 (1990) (in Russian).

[15] T. M. Osipchuk, *Analytic conditions of local linear convexity in $\mathbb{H}^n$*, Zb. prats of the Inst. of Math. of NASU **3**, no. 3, 244–254 (2006) (in Ukrainian).

[16] T.M. Osipchuk, *Analytical conditions of local linear convexity in the space $\mathbb{H}^n_{\alpha,\beta}$*, Zb. prats of the Inst. of Math. of NASU **10**:4–5, 301–305 (2013).

[17] T.M. Osipchuk, Yu.B. Zelinskii, M.V. Tkachuk, *Analytical conditions of locally general convexity in $C^n_{p,q}$*, Zb. prats of the Inst. of Math. of NASU **7**:2, 393–401 (2010) (in Ukrainian).

[18] R. Delanghe, F. Sommen and V. Soücek, *Clifford Algebra and Spinorvalued functions: a function theory for the Dirac-operator*, Dordrecht.: Kluwer Academic Publishers, Mathematics and Its Applications, **53** (1992), 322 p.

[19] D. K. Fadeev, *Lectures on Algebra*, M.: Nauka (1984) 416 p (in Russian).

[20] K. Leyhtveys, *Convex sets*: Transl. from German. M.: Nauka (1985) (in Russian).

Presented partially, by Tetiana Osipchuk, online,
during the Hypercomplex Seminar, Nov. 11th, 2021
Watch: T. Osipchuk, ON LINEAL CONVEXITY...

Tetiana M. Osipchuk
Institute of Mathematics of the National Academy of Sciences of Ukraine
Tereshchenkivska str. 3, UA-01004, Kyiv, Ukraine
e-mail: osipchuk.tania@gmail.com

O wypukłości liniowej uogólnionej do algebr Clifforda

S t r e s z c z e n i e

Pojęcie obszarów liniowo osiągalnych z zewnątrz w skończenie wymiarowej przestrzeni zespolonej i niektóre ich własności są uogólniane na skończenie wymiarową przestrzeń $\mathcal{C}\ell^m_{p,q}$, $m \geq 2$, będącą iloczynem kartezjańskim m uniwersalnych algebr Clifforda $\mathcal{C}\ell_{p,q}$ nad ciałem liczb rzeczywistych. Mianowicie, dla dowolnego ciągu $d_1 d_2 \dots d_m$ uzyskano warunki konieczne i wystarczające lokalnej $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-liniowej osiągalności z zewnątrz obszarów o gładkim brzegu, gdzie $d_j \in \{L, R\}$, $j = \overline{1, m}$. Warunki te są uogólnieniem dobrze znanych warunków lokalnej liniowej osiągalności z zewnątrz obszaru o gładkim brzegu, uzyskanych przez B. Zinowiewa.

Słowa kluczowe: Zbiór wypukły, zbiór liniowo osiągalny z zewnątrz, zbiór $(\mathcal{C}\ell_{p,q}, d_1 d_2 \dots d_m)$-liniowo osiągalny z zewnątrz, algebra Clifforda, forma liniowa, forma kwadratowa, forma różniczkowa, pochodna formalna.