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Dedicated to the memory of
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MEAN VALUE THEOREMS FOR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Summary

We prove a mean value theorem that characterizes continuous weak solutions of homogeneous linear partial differential equations with constant coefficients in Euclidean domains. In this theorem the mean value of a smooth function with respect to a complex Borel measure on an ellipsoid of special form is equal to some linear combination of its partial derivatives at the center of this ellipsoid. The main result of the paper generalizes a well-known Zalcman's theorem.

Keywords and phrases: mean value, linear partial differential operator, weak solution, Fourier-Laplace transform, distribution

1. Introduction

Let $P(D)$ be a linear partial differential operator with constant coefficients in the Euclidean space $\mathbb{R}^n$, $n \geq 1$, and let μ be a complex Borel measure supported in the closed unit ball B of $\mathbb{R}^n$. Zalcman [1] proved the equivalence of the following assertions: (a) for any domain $G \subset \mathbb{R}^n$ and for any complex-valued function $u \in C(G)$,

$$\int u(\mathbf{x} + r\mathbf{t})\, d\mu(\mathbf{t}) = 0$$

for all $\mathbf{x} \in G$ and $r \in (0,\ \mathrm{dist}(\mathbf{x}, \partial G))$ if and only if u is a weak solution of the equation $P(D)f = 0$ in G; (b) the operator $P(D)$ is homogeneous and the functional $F_\mu(\varphi) := \int \varphi(\mathbf{t})\, d\mu(\mathbf{t})$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, in the space $\mathcal{E}'(\mathbb{R}^n)$ is represented in the form $F_\mu = P(D)T$ for some distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B with $\hat{T}(\mathbf{0}) \neq 0$, where $\hat{T}$ is the Fourier–Laplace transform of T. This result was the first general mean value theorem for solutions of linear partial differential equations, which contains the classical Gauss characterization of harmonic functions by spherical means, the Morera–Carleman characterization of analytic functions of a complex variable by zero integrals $\int f(z)\, dz$ over circles, and some other concrete mean value theorems as special cases. The first author [2] generalized Zalcman's result for the case of quasihomogeneous operators and applied this generalization to the study of removable singularities of solutions of the equation $P(D)f = 0$ with quasihomogeneous semielliptic operator $P(D)$ [3].

On the other hand, the second author studied classes of smooth functions defined in a disk $B(0, R) := \{z \in \mathbb{C} : |z| < R\}$ that satisfy the condition

$$\sum_{p=s}^{m-1} \frac{r^{2p+2}}{(2p+2)(p-s)!p!} \partial^{p-s}\bar{\partial}^p f(z) = \frac{1}{2\pi} \iint_{|\zeta - z| \le r} f(\zeta)(\zeta - z)^s d\xi d\eta, \tag{1}$$

where $R > 0$, $s \in \mathbb{N}_0$, $m \in \mathbb{N}$, $s < m$, $z = x + iy$, $\zeta = \xi + i\eta$ $(x, y, \xi, \eta \in \mathbb{R})$, i is the imaginary unit,

$$\partial f = \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

She proved [4] as a special case of more general result that each function $f \in C^{2(m-1)-s}(B(0, R))$ satisfying this condition for all $r \in (0,\ R)$ and $z \in B(0, R - r)$ is a solution of the equation $\partial^{m-s}\bar{\partial}^m f = 0$.

In the present paper we prove a mean value theorem of Zalcman type that contains all the mentioned results as special cases.

2. Formulation of the main result

Let $n \in \mathbb{N} := \{1, 2, \ldots\}$ and let $\mathbf{M} = (M_1, \ldots, M_n)$ be a vector with positive integer components, $|\mathbf{M}| = M_1 + \ldots + M_n$. To each polynomial $P = P(\mathbf{z})$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, with complex-valued coefficients and to each $r > 0$ we assign the differential operator $P(r^{\mathbf{M}}D)$, in which z_k, $k = 1, \ldots, n$, is replaced by $-ir^{M_k}\partial/\partial x_k$. If $\mathbf{M} = (1, \ldots, 1)$, then $P(r^{\mathbf{M}}D) =: P(rD)$. A polynomial $P(\mathbf{z})$ (an operator $P(D) := P(1^{\mathbf{M}}D)$) is said to be $\mathbf{M}$-homogeneous if there is an $l \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ such that $P(\mathbf{z}) \equiv \sum_{\mathbf{k}} a_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$, where $\mathbf{z}^{\mathbf{k}} := z_1^{k_1} \ldots z_n^{k_n}$ and the sum is taken over the set of all multiindices $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ with $|\mathbf{kM}| := k_1M_1 + \ldots + k_nM_n = l$. For any polynomial $P(\mathbf{z}) \equiv \sum_{\mathbf{k}} a_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$ we denote by $\deg_{\mathbf{M}} P$ the number $\sup |\mathbf{kM}|$, where the supremum is taken over all multiindices $\mathbf{k} \in \mathbb{N}_0$ with $a_{\mathbf{k}} \neq 0$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$

and $r > 0$ we use the following notation: $r^{\mathbf{M}}\mathbf{x} := (r^{M_1}x_1, \ldots .r^{M_n}x_n)$, $B_{\mathbf{M}}(\mathbf{x}, r) := \{\mathbf{x} + r^{\mathbf{M}}\mathbf{t} : \mathbf{t} \in \mathbb{R}^n, |\mathbf{t}| \le 1\}$. If $\mathbf{M} = (1, \ldots, 1)$, then $\deg_{\mathbf{M}} P =: \deg P$, $B_{\mathbf{M}}(\mathbf{x}, r) =: B(\mathbf{x}, r)$. Recall that the Fourier–Laplace transform of a distribution $f \in \mathcal{E}'(\mathbb{R}^n)$ is defined by the formula $\hat{f}(\mathbf{z}) := f(e^{-i(\mathbf{x}\cdot\mathbf{z})})$, where $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{z} = x_1z_1 + \ldots + x_nz_n$, and the distribution f acts on the function $e^{-i(x\cdot z)}$ in x. As usual, δ is the Dirac measure, i.e., the unit measure concentrated at the origin.

Let μ be a complex Borel measure supported in $B := B(\mathbf{0}, 1)$ and let $P = P(\mathbf{z})$ and $Q = Q(\mathbf{z})$ be polynomials with complex-valued coefficients ($\mathbf{z} \in \mathbb{C}^n$). Denote by F_μ the functional corresponding to the measure μ in the space $\mathcal{E}'(\mathbb{R}^n)$, i.e., $F_\mu(\varphi) := \int \varphi(\mathbf{t})\, d\mu(\mathbf{t})$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\hat{\mu}(\mathbf{z}) := \hat{F_\mu}(\mathbf{z})$.

Definition 1. *We say that a triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $P(D)f = 0$ if for any domain $G \subset \mathbb{R}^n$ and for any function $u \in C(G)$ the following conditions are equivalent:*
(a) u is a weak solution of the equation $P(D)f = 0$ in G;
(b) for all $\varphi \in C_0^\infty(G)$ and $r > 0$ such that $\operatorname{supp}\varphi + B_{\mathbf{M}}(\mathbf{0}, r) \subset G$ *we have*

$$\int_G u(\mathbf{x})\Big(\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) - Q(-r^{\mathbf{M}}D)\varphi(\mathbf{x})\Big)\, d\mathbf{x} = 0.$$

Here, as usual,

$$\operatorname{supp}\varphi + B_{\mathbf{M}}(\mathbf{0}, r) := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \operatorname{supp}\varphi,\ \ \mathbf{y} \in B_{\mathbf{M}}(\mathbf{0}, r)\}.$$

The main result of this paper is the following theorem.

Theorem 1. *A triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $P(D)f = 0$ if and only if the polynomial P is $\mathbf{M}$-homogeneous and there is a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B such that $\hat{T}(\mathbf{0}) \ne 0$ and $F_\mu = P(-D)T + Q(-D)F_\delta$.*

3. Auxiliary results

The proof of Theorem 1 is essentially based on Zalcman's arguments [1] and uses the following lemmas.

Lemma 1 [5, Theorem 7.3.2]. *Suppose that $f \in \mathcal{E}'(\mathbb{R}^n)$ and $P(D)$ is a linear differential operator with constant coefficients. The equation $P(D)u = f$ has a distributional solution $u \in \mathcal{E}'(\mathbb{R}^n)$ if and only if $\hat{f}(\mathbf{z})/P(\mathbf{z})$ is an entire function. In this case the solution is determined uniquely, and the closure of the convex hull of the support of the distribution u coincides with that of the distribution f.*

Suppose that polynomials P_k, $k \in \mathbb{N}_0$, are given. If a function u satisfies the equalities $P_k(D)u = 0$ in $\mathbb{R}^n$ for each $k \in \mathbb{N}_0$ and

$$u(\mathbf{x}) \equiv g(\mathbf{x})e^{-i(\mathbf{z}\cdot\mathbf{x})} \tag{2}$$

for some polynomial $g(\mathbf{x})$ and $\mathbf{z} \in \mathbb{C}^n$, then we say that u is an exponential solution of the system $P_k(D)f = 0$, $k \in \mathbb{N}_0$.

Lemma 2 [5, Lemma 7.3.7]. *Suppose that $P(D)$ is a linear differential operator with constant coefficients. If $\nu \in \mathcal{E}'(\mathbb{R}^n)$ is a distribution such that $\nu(u) = 0$ for each exponential solution u of the equation $P(D)f = 0$, then $\hat{\nu}(\mathbf{z})/P(-\mathbf{z})$ is an entire function.*

Lemma 3 [6, Theorem 7.6.14]. *Suppose that G is a convex domain in $\mathbb{R}^n$, $q \in \mathbb{N}_0$, and $P_k(D)$, $k = 0, \dots, q$, is a finite set of linear differential operators with constant coefficients. Then each continuous weak solution of the system $P_k(D)f = 0$, $k = 0, \dots, q$, in G can be represented in the form of the limit of some sequence of finite linear combinations of exponential solutions of this system, uniformly converging on compact subsets of G.*

Lemma 4 [1, Theorem 3], [2, Lemma 1]. *Suppose that polynomials $P(\mathbf{z})$ and $P_j(\mathbf{z})$, $j \in \mathbb{N}_0$, are such that, for each $j \in \mathbb{N}_0$, either $P_j(\mathbf{z})$ is an $\mathbf{M}$-homogeneous polynomial with $\deg_{\mathbf{M}} P_j = j$ or $P_j(\mathbf{z}) \equiv 0$ ($\mathbf{z} \in \mathbb{C}^n$). Moreover, let $P_j(\mathbf{z}) \not\equiv 0$ for at least one $j \in \mathbb{N}_0$. The system of differential equations $P_j(D)f = 0$, $j \in \mathbb{N}_0$, is equivalent to the equation $P(D)f = 0$ is and only if each of the polynomials $P_j(\mathbf{z})$, $j \in \mathbb{N}_0$, is divisible by the polynomial $P(\mathbf{z})$ and for some number $k \in \mathbb{N}_0$ the polynomial $P_k(\mathbf{z})$ coincides with the polynomial $P(\mathbf{z})$ up to a nonzero constant factor.*

4. Proof of Theorem 1

Suppose that $\mathbf{M} = (M_1, \dots, M_n)$ ($n \geq 1$) is a vector with positive integer components, μ is a complex Borel measure supported in B, $Q(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^n$) is a polynomial, and u is a function of the form (2) in $\mathbb{R}^n$ satisfying the condition

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = Q(r^{\mathbf{M}}D)u(\mathbf{x}) \tag{3}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Let us choose a point $\mathbf{x} \in \mathbb{R}^n$ and expand the function u in the Taylor series around $\mathbf{x}$. Collecting $\mathbf{M}$-homogeneous polynomials in this series, we obtain

$$u(\mathbf{x} + \mathbf{y}) = \sum_{j=0}^{\infty} U_j(\mathbf{y}), \tag{4}$$

where

$$U_j(\mathbf{y}) := \sum_{|\mathbf{kM}|=j} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} u(\mathbf{x}) \mathbf{y}^{\mathbf{k}},$$

$$\mathbf{k}! := k_1! \dots k_n!, \quad \mathbf{y}^{\mathbf{k}} := y_1^{k_1} \dots y_n^{k_n}, \quad \partial^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}.$$

Similarly, we represent the polynomial Q in the form of the finite sum of $\mathbf{M}$-homogeneous polynomials:

$$Q(\mathbf{y}) = \sum_{j=0}^{d} Q_j(\mathbf{y}), \tag{5}$$

where $d = \deg_{\mathbf{M}} Q$, $Q_j(\mathbf{z})$ is either an $\mathbf{M}$-homogeneous polynomial with $\deg_{\mathbf{M}} Q_j = j$ or $Q_j(\mathbf{z}) \equiv 0$, $Q_j(\mathbf{z}) \equiv 0$ for all $j > d$ ($\mathbf{z} \in \mathbb{C}^n$). The series in (4) converges to $u(\mathbf{x}+\mathbf{y})$ uniformly on compact sets in $\mathbb{R}^n$. Let us choose an arbitrary $r > 0$ and set $\mathbf{y} = r^{\mathbf{M}}\mathbf{t}$, where $\mathbf{t} \in B$. Since the series in (4) converges uniformly, we can integrate both sides of the resultant relation with respect to the measure μ term by term. This yields

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = \sum_{j=0}^{\infty} U_j(r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = \sum_{j=0}^{\infty} r^j (R_j(D)u)(\mathbf{x}), \tag{6}$$

where

$$R_j(\mathbf{z}) = \sum_{|\mathbf{kM}|=j} (\mathbf{k}!)^{-1}(i\mathbf{z})^{\mathbf{k}} \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}), \quad j \in \mathbb{N}_0, \quad \mathbf{z} \in \mathbb{C}^n. \tag{7}$$

Let $P_j(\mathbf{z}) := R_j(\mathbf{z}) - Q_j(\mathbf{z})$, $j \in \mathbb{N}_0$. Then it follows from (4)-(6) that

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) - Q(r^{\mathbf{M}}D)u(\mathbf{x}) = \sum_{j=0}^{\infty} r^j (P_j(D)u)(\mathbf{x}). \tag{8}$$

Since the condition (3) holds for any $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, we have $P_j(D)u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$.

Let G be a domain in $\mathbb{R}^n$ and let $\varphi \in C_0^\infty(G)$. Take $\mathbf{x} \in G$ and $r > 0$ such that $B_{\mathbf{M}}(\mathbf{x}, r) \subset G$. By the Taylor formula with reminder in integral form, for each $l \in \mathbb{N}$ and for all $\mathbf{y} \in B_{\mathbf{M}}(\mathbf{0}, r)$, we have

$$\varphi(\mathbf{x} + \mathbf{y}) = \sum_{|\mathbf{k}|<l} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} \varphi(\mathbf{x}) \mathbf{y}^{\mathbf{k}} + l \int_0^1 (1-s)^{l-1} \Big(\sum_{|\mathbf{k}|=l} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} \varphi(\mathbf{x} + s\mathbf{y}) \mathbf{y}^{\mathbf{k}} \Big) ds.$$

By setting $\mathbf{y} = -r^{\mathbf{M}}\mathbf{t}$, $\mathbf{t} \in B$, and rearranging the terms, we obtain

$$\varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t}) = \sum_{j=0}^{p} r^j \Big(\sum_{|\mathbf{kM}|=j} (-1)^{|\mathbf{k}|} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} \varphi(\mathbf{x}) \mathbf{t}^{\mathbf{k}} \Big) + V_p(r, \mathbf{x}, \mathbf{t}), \tag{9}$$

where $p = p(l)$ is the largest of numbers such that $|\mathbf{kM}| \le p$ implies $|\mathbf{k}| < l$ for any multiindex $\mathbf{k}$; $V_p(r, \mathbf{x}, \mathbf{t}) = o(r^p)$ as $r \to 0$ uniformly in $\mathbf{x} \in \operatorname{supp} \varphi$ and $\mathbf{t} \in B$. It is clear that (9) holds for each $p \in \mathbb{N}_0$. Integrating both sides of (9) with respect to the measure μ, we obtain

$$\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\,d\mu(\mathbf{t}) = \sum_{j=0}^{p} r^j \Big(\sum_{|\mathbf{kM}|=j} (-1)^{|\mathbf{k}|}(\mathbf{k}!)^{-1}\partial^{\mathbf{k}}\varphi(\mathbf{x}) \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t})\Big) + W_p(r,\mathbf{x}),$$

or

$$\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\,d\mu(\mathbf{t}) = \sum_{j=0}^{p} r^j (R_j(-D)\varphi)(\mathbf{x}) + W_p(r,\mathbf{x}), \tag{10}$$

where

$$(R_j(-D)\varphi)(\mathbf{x}) = \sum_{|\mathbf{kM}|=j} (-1)^{|\mathbf{k}|}(\mathbf{k}!)^{-1}(-1)^{|\mathbf{k}|}\partial^{\mathbf{k}}\varphi(\mathbf{x}) \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}),\ j \in \mathbb{N}_0.$$

$W_p(r,\mathbf{x}) = o(r^p)$ as $r \to 0$ uniformly in $\mathbf{x} \in \operatorname{supp}\varphi$.

Now we assume that a function $u \in C(G)$ satisfies the condition (b) of Definition 1. Then we have from (10) (for sufficiently small $r > 0$)

$$0 = \int_G u(\mathbf{x})\Big(\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\,d\mu(\mathbf{t}) - Q(-r^{\mathbf{M}}D)\varphi(\mathbf{x})\Big)\,d\mathbf{x} = \sum_{j=0}^{p} r^j \int_G u(\mathbf{x})((R_j - Q_j)(-D)\varphi)(\mathbf{x})\,d\mathbf{x} + o(r^p),$$

or

$$0 = \sum_{j=0}^{p} r^j \int_G u(\mathbf{x})(P_j(-D)\varphi)(\mathbf{x})\,d\mathbf{x} + o(r^p) \ \text{ as } \ r \to 0, \tag{11}$$

where $P_j(-D) = R_j(-D) - Q_j(-D)$, $j \in \mathbb{N}_0$. Suppose that at least one of the polynomials $\{P_j(\mathbf{z})\}_{j\in\mathbb{N}_0}$ does not vanish identically and p is the least number such that $P_p(\mathbf{z}) \not\equiv 0$. Dividing both side of (11) by r^p and letting $r \to 0$, we obtain $\int_G u(\mathbf{x})(P_p(-D)\varphi)(\mathbf{x})\,d\mathbf{x} = 0$. Then, proceeding by induction, we have

$$\int_G u(\mathbf{x})(P_j(-D)\varphi)(\mathbf{x})\,d\mathbf{x} = 0 \quad \forall j \in \mathbb{N}_0.$$

If all the polynomials $\{P_j(\mathbf{z})\}_{j\in\mathbb{N}_0}$ are identically zero, then the last assertion is obvious.

Since the function φ was an arbitrary function from $C_0^\infty(G)$ in our arguments, we have that u is a weak solution of the system

$$P_j(D)f = 0, \quad j \in \mathbb{N}_0. \tag{12}$$

Conversely, if u is a weak solution of the system (12) in G, then u satisfies the condition (b) of Definition 1. For exponential solutions this was justified by formula (8). The general case follows from Lemma 3 and the Hilbert Basis Theorem [7], which implies that there is a $j_0 \in \mathbb{N}_0$ such that the system (12) is equivalent to the finite system of differential equations $P_j(D)f = 0$, $j = 0, 1, \ldots, j_0$.

To complete the proof of Theorem 1 we should investigate conditions of equivalence of the system (12) and the equation $P(D)f = 0$. The Fourier–Laplace transform $\hat{\mu}(\mathbf{z})$ of the measure μ is an entire function and its Taylor series around the point $\mathbf{z} = 0$ converges absolutely and uniformly on each compact set in $\mathbb{C}^n$. Therefore, by arranging of $\mathbf{M}$-homogeneous polynomials $R_j(-\mathbf{z})$ in this series, we obtain a series that uniformly converges to $\hat{\mu}(\mathbf{z})$ on compact sets in $\mathbb{C}^n$ as follows:

$$\hat{\mu}(\mathbf{z}) = \int e^{-i(\mathbf{z}\cdot\mathbf{t})} d\mu(\mathbf{t}) = \sum_{j=0}^{\infty} R_j(-\mathbf{z}),$$

where the sequence of polynomials $\{R_j(\mathbf{z})\}_{j\in\mathbb{N}_0}$ is defined by (7). Suppose that the triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $P(D)f = 0$. Then this equation is equivalent to the system (12). If $P(\mathbf{z}) \equiv 0$, then $R_j(\mathbf{z}) \equiv Q_j(\mathbf{z})$ for all $j \in \mathbb{N}_0$ and consequently

$$\hat{\mu}(\mathbf{z}) \equiv \sum_{j=0}^{\infty} Q_j(-\mathbf{z}) = Q(-\mathbf{z}).$$

Hence $F_\mu = Q(-D)F_\delta$, which is possible if only if $\deg Q = 0$. Now consider the case $P(\mathbf{z}) \not\equiv 0$. Then there is a number $p \in \mathbb{N}_0$ such that $P_p(\mathbf{z}) \not\equiv 0$. Since the divisors of an $\mathbf{M}$-homogeneous polynomial are also $\mathbf{M}$-homogeneous polynomials, then we have from Lemma 4 that the polynomial $P(\mathbf{z})$ is $\mathbf{M}$-homogeneous. It follows from the fact that the triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $P(D)f = 0$ and from Lemma 2 that $S(\mathbf{z}) := (\hat{\mu}(\mathbf{z}) - Q(-\mathbf{z}))/P(-\mathbf{z})$ is an entire function whence Lemma 1 implies that there is a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B such that

$$F_\mu = P(-D)T + Q(-D)F_\delta. \tag{13}$$

By applying the Fourier–Laplace transform to both sides of (13), we have $\hat{\mu}(\mathbf{z}) \equiv P(-\mathbf{z})\hat{T}(\mathbf{z}) + Q(-\mathbf{z})$. This means that $S(\mathbf{z}) \equiv T(\mathbf{z})$ and we derive the condition $\hat{T}(\mathbf{0}) \neq 0$ from the fact that an entire function can be uniquely represented by a series of $\mathbf{M}$-homogeneous polynomials uniformly convergent on compact subsets of $\mathbb{C}^n$.

Thus we justify the 'only if' part in Theorem 1. To prove the 'if' part of this theorem suppose that $P(\mathbf{z})$ is an $\mathbf{M}$-homogeneous polynomial, $m = \deg_{\mathbf{M}} P$, and T is a distribution supported in B satisfying (13). In this case $\hat{T}(\mathbf{0}) \neq 0$ need not hold. Let u be an exponential solution of the equation $P(D)f = 0$ in $\mathbb{R}^n$. If $\hat{T}(\mathbf{z}) = \sum_{j=0}^{\infty} T_j(-\mathbf{z})$ is the Taylor series of the entire function $\hat{T}$ around the point $\mathbf{z} = 0$ arranged in $\mathbf{M}$-homogeneous polynomials ($\deg_{\mathbf{M}} T_j = j$ or $T_j(\mathbf{z}) \equiv 0$), then, by comparing the equalities

$$\hat{\mu}(\mathbf{z}) - Q(-\mathbf{z}) = P(-\mathbf{z})T(\mathbf{z}), \qquad \hat{\mu}(\mathbf{z}) - Q(-\mathbf{z}) = \sum_{j=0}^{\infty} P_j(-\mathbf{z}),$$

and (8), we see that $P_j(\mathbf{z}) \equiv 0$ for all $j < m$, $P_{j+m}(\mathbf{z}) \equiv P(\mathbf{z})T_j(\mathbf{z})$ for all $j \in \mathbb{N}_0$, and

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) - Q(r^{\mathbf{M}}D)u(\mathbf{x}) = \sum_{j=0}^{\infty} r^{j+m}(P(D)T_j(D)u(\mathbf{x}) = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$. The case of arbitrary continuous weak solutions of the equation $P(D)f = 0$ is reduced to the case of exponential solutions by applying Lemma 3, the Hilbert Basis Theorem, and integration by parts. The proof of Theorem 1 is completed.

5. Discussion of Theorem 1

Let $Q(\mathbf{z}) \equiv 0$ in Theorem 1. Then the condition (b) of Definition 1 is rewritten in the form

$$\int_G u(\mathbf{x})\Big(\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t})\Big)\, d\mathbf{x} = 0$$

for all $\varphi \in C_0^\infty(G)$ and $r > 0$ such that $\operatorname{supp}\varphi + B_{\mathbf{M}}(\mathbf{0}, r) \subset G$ whence

$$\int_G \Big(\int u((\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\varphi(\mathbf{x})\, d\mu(\mathbf{t})\Big)\, d\mathbf{x} = 0$$

for all such φ and r. It follows from the Fubini theorem that

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = 0. \tag{14}$$

This means that the condition (b) of Theorem 1 is satisfied if and only if (14) holds for all $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$ such that $B_{\mathbf{M}}(\mathbf{x}, r) \subset G$. Hence, for $Q(\mathbf{z}) \equiv 0$, Theorem 1 coincides with Theorem 2 from [2], which generalizes the mentioned Zalcman's result [1, Theorem 4] corresponding to the case $Q(\mathbf{z}) \equiv 0$ and $\mathbf{M} = (1, \ldots, 1)$ in Theorem 1.

Now consider the case $n = 2$, $\mathbf{M} = (1, 1)$, and rewrite (1) in the form

$$Q(rD) = \int_B f(z + rt)\, d\mu(t), \tag{15}$$

where G is a domain in $\mathbb{C}$, $f \in C^{2m-2-s}(G)$, $z \in G$, $r > 0$, $B(z, r) \subset G$,

$$Q(z_1, z_2) = \sum_{p=s}^{m-1} (2^{2p-s}\pi(p+1)(p-s)!p!)^{-1}(iz_1 + z_2)^{p-s}(iz_1 - z_2)^p,$$

$d\mu(t) = t^s dt_1 dt_2$, $t = t_1 + it_2$, $t_1, t_2 \in \mathbb{R}$. Introduce the variables $w_1 = iz_1 + z_2$ and $w_2 = iz_1 - z_2$. Then $z_1 = -i(w_1 + w_2)/2$, $z_2 = (w_1 - w_2)/2$, and the Fourier–Laplace transform of μ can be expressed as follows:

$$\hat{\mu}(z_1, z_2) = \int_B e^{-i(z_1t_1+z_2t_2)} t^s dt_1 dt_2$$
$$= \int_B e^{-(w_1+w_2)t_1/2 - i(w_1-w_2)t_2/2} t^s dt_1 dt_2$$
$$= \int_B e^{-w_1(t_1-it_2)/2 - w_2(t_1+it_2)/2} t^s dt_1 dt_2$$
$$= \sum_{k,l=0}^{\infty} (-2)^{k+l}(k!l!)^{-1} w_1^k w_1^l \int_B (t_1 - it_2)^k (t_1 + it_2)^{l+s} dt_1 dt_2$$
$$= \sum_{p=s}^{\infty} (-2)^{2p-s} s((p-s)!p!)^{-1}(iz_1 + z_2)^{p-s}(iz_1 - z_2)^p \int_B |t|^{2p} dt_1 dt_2$$
$$= \sum_{p=s}^{\infty} (-2)^{2p-s}((p-s)!p!)^{-1}(iz_1 + z_2)^{p-s}(iz_1 - z_2)^p 2\pi(2p+2)^{-1}.$$

This chain of equalities shows that there is a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B such that $\hat{T}(0,0) \neq 0$ and

$$\hat{\mu}(z_1, z_2) - Q(-z_1, -z_2) \equiv (-2)^{-(2m-s)}(iz_1 + z_2)^{m-s}(iz_1 - z_2)^m \hat{T}(z_1, z_2).$$

Theorem 1 implies that the triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $\partial^{m-s}\bar{\partial}^m f = 0$. Since the differential operator $\partial^{m-s}\bar{\partial}^m$ is elliptic and consequently its distributional and classical solutions coincide, then we show that a function $f \in C^{2(m-1)-s}(G)$ satisfies the condition (15) for all $z \in G$ and $r \in (0, \operatorname{dist}(z, \partial G))$ if and only if f is a solution of the equation $\partial^{m-s}\bar{\partial}^m f = 0$ in G.

Note that the conditions in the 'only if' part of the last assertion can be essentially weakened. Namely, let $m \in \mathbb{N}$, $s \in \mathbb{N}_0$, $s < m$, and let

$$J_{s+1}(z) := \left(\frac{z}{2}\right)^{s+1} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(s+p+2)} \left(\frac{z}{2}\right)^{2p} \qquad (z \in \mathbb{C})$$

be the Bessel function. For $r > 0$ denote by Z_r the set of all zeros of the entire function

$$g_{s,m,r}(z) := \frac{J_{s+1}(zr)}{(zr)^{s+1}} - \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}}$$

belonging to $\mathbb{C} \setminus \{0\}$. Let r_1, r_2, R be positive numbers. The following result was proved in [4]: (a) if $R > r_1 + r_2$, $Z_{r_1} \cap Z_{r_2} = \varnothing$, $f \in C^{2m-2-s}(B(0,R))$, and the condition (1) holds for all $r \in \{r_1, r_2\}$ and $z \in B(0, R-r)$, then f belongs to the class $C^\infty(B(0,R))$ and satisfies the differential equation $\partial^{m-s}\bar{\partial}^m f = 0$; (b) if $\max\{r_1, r_2\} < R < r_1 + r_2$ or $Z_{r_1} \cap Z_{r_2} \neq \varnothing$, then there exists a function $f \in C^\infty(B(0,R))$ satisfying the condition (1) for all $r \in \{r_1, r_2\}$ and $z \in B(0, R-r)$ that is not a solution of this equation in $B(0,R)$.

In the case $m = 1$ and $s = 0$ assertions (a) and (b) coincide with assertions (1) and (4) of Theorem 5.4 from [8, p. 399] for $n = 2$, respectively, where the local version of the classical Delsarte's two-radii theorem [9] characterizing harmonic functions in $\mathbb{R}^n$ is presented.

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TWIERDZENIE O WARTOŚCI ŚREDNIEJ DLA ROZWIĄZAŃ LINIOWYCH RÓWNAŃ RÓŻNICZKOWYCH O POCHODNYCH CZĄSTKOWYCH O STAŁYCH WSPÓŁCZYNNIKACH

S t r e s z c z e n i e

Wykazujemy twierdzenie o wartości średniej, które charakteryzuje ciągłe słabe rozwiązania jednorodnych liniowych równań różniczkowych cząstkowych o stałych współczynnikach w obszarach euklidesowych. W twierdzeniu tym wartość średnia funkcji gładkiej względem zespolonej miary borelowskiej na pewnej elipsoidzie specjalnej postaci jest równa pewnej kombinacji liniowej jej pochodnych cząstkowych w środku tej elipsoidy. Główny wynik pracy uogólnia znane twierdzenie Zalcmana

Słowa kluczowe: wartość średnia liniowego operatora różniczkowego cząstkowego, słabe rozwiązanie, transformata Fouriera-Laplace'a, dystrybucja