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Dedicated to the memory of
Professor Yurii B. Zelinskii

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LEBESGUE-TYPE INEQUALITIES FOR THE FOURIER SUMS ON CLASSES OF GENERALIZED POISSON INTEGRALS

Summary

For functions from the set of generalized Poisson integrals $C^{\alpha,r}_{\beta}L_p$, $1 \leq p < \infty$, we obtain upper estimates for the deviations of Fourier sums in the uniform metric in terms of the best approximations of the generalized derivatives $f^{\alpha,r}_{\beta}$ of functions of this kind by trigonometric polynomials in the metric of the spaces L_p. Obtained estimates are asymptotically best possible.

Keywords and phrases: Lebesgue-type inequalities, Fourier sums, generalized Poisson integrals, best approximations

Let L_p, $1 \leq p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt\Big)^{\frac{1}{p}}$; L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm $\|f\|_\infty = \operatorname{ess\,sup}\limits_t |f(t)|$; C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Denote by $C^{\alpha,r}_{\beta}L_p$, $\alpha > 0$, $r > 0$, $\beta \in \mathbb{R}$, $1 \leq p \leq \infty$, the set of all 2π–periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [1, p. 133])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt,\ a_0 \in \mathbb{R},\quad \varphi \perp 1, \tag{1}$$

where $\varphi \in L_p$ and $P_{\alpha,r,\beta}(t)$ are fixed generated kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\big(kt - \frac{\beta\pi}{2}\big), \quad \alpha, r > 0, \quad \beta \in \mathbb{R}. \tag{2}$$

The kernels $P_{\alpha,r,\beta}$ of the form (2) are called generalized Poisson kernels. For $r = 1$ and $\beta = 0$ the kernels $P_{\alpha,r,\beta}$ are usual Poisson kernels of harmonic functions.

If the functions f and φ are related by the equality (1), then function f in this equality is called generalized Poisson integral of the function φ. The function φ in equality (1) is called as generalixed derivative of the function f and is denoted by $f_\beta^{\alpha,r}$.

The set of functions f from $C_\beta^{\alpha,t}L_p$, $1 \le p \le \infty$, such that $f_\beta^{\alpha,r} \in B_p^0$, where

$$B_p^0 = \{\varphi :\ ||\varphi||_p \le 1,\ \varphi \perp 1\},\ 1 \le p \le \infty,$$

we will denote by $C_{\beta,p}^{\alpha,r}$.

Let $E_n(f)_{L_p}$ be the best approximation of the function $f \in L_p$ in the metric of space L_p, $1 \le p \le \infty$, by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_p} = \inf_{t_{n-1}} \|f - t_{n-1}\|_{L_p}.$$

Let $\rho_n(f;x)$ be the following quantity

$$\rho_n(f;x) := f(x) - S_{n-1}(f;x), \tag{3}$$

where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of order $n-1$ for a function f.

Least upper bounds of the quantity $\|\rho_n(f;\cdot)\|_C$ over the classes $C_{\beta,p}^{\alpha,r}$, we denote by $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$, i.e.,

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \sup_{f \in C_{\beta,p}^{\alpha,r}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_C,\ r > 0,\ \alpha > 0,\ 1 \le p \le \infty. \tag{4}$$

Asymptotic behaviour of the quantities $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ of the form (4) was studied in [2]–[10].

In [11]–[13] it was found the analogs of the Lebesgue inequalities for functions $f \in C_\beta^{\alpha,r}L_p$ in the case $r \in (0,1)$ and $p = \infty$, and also in the case $r \ge 1$ and $1 \le p \le \infty$, where the estimates for the deviations $\|f(\cdot) - S_{n-1}(f;\cdot)\|_C$ are expressed in terms of the best approximations $E_n(f_\beta^{\alpha,r})_{L_p}$. Namely, in [11] it is proved that the following best possible inequalitiy holds

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le \Big(\frac{4}{\pi^2}\ln n^{1-r} + O(1)\Big)e^{-\alpha n^r} E_n(f_\beta^{\alpha,r})_{L_\infty}, \tag{5}$$

where $O(1)$ is a quantity uniformly bounded with respect to n, β and $f \in C_\beta^{\alpha,r}L_\infty$.

The present paper is a continuation of [11], [12], and is devoted to getting asymptotically best possible analogs of Lebesgue-type inequalities on the sets $C^{\alpha,r}_{\beta}L_p$, $r \in (0,1)$ and $p \in [1,\infty)$. This case was not considered yet. Let formulate the results of the paper.

By $F(a,b;c;d)$ we denote Gauss hypergeometric function

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

$$(x)_k := \frac{x}{2}\Big(\frac{x}{2}+1\Big)\Big(\frac{x}{2}+2\Big)...\Big(\frac{x}{2}+k-1\Big).$$

For arbitrary $\alpha > 0$, $r \in (0,1)$ and $1 \le p \le \infty$ we denote by $n_0 = n_0(\alpha, r, p)$ the smallest integer n such that

$$\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \le \begin{cases} \frac{1}{14}, & p = 1, \\ \frac{1}{(3\pi)^3} \cdot \frac{p-1}{p}, & 1 < p < \infty, \\ \frac{1}{(3\pi)^3}, & p = \infty, \end{cases} \tag{6}$$

where $\chi(p) = p$ for $1 \le p < \infty$ and $\chi(p) = 1$ for $p = \infty$.

The following statement holds.

Theorem 1. *Let $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then in the case $1 < p < \infty$ for any function $f \in C^{\alpha,r}_{\beta}L_p$ and $n \ge n_0(\alpha, r, p)$, fthe following inequality is true:*

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma_{n,p}\Bigg(\Bigg(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Bigg)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Bigg)\Bigg) E_n(f^{\alpha,r}_{\beta})_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{7}$$

where $F(a,b;c;d)$ is Gauss hypergeometric function, and in the case $p = 1$ for any function $f \in C^{\alpha,r}_{\beta}L_1$ and $n \ge n_0(\alpha, r, 1)$, the following inequality is true:

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le e^{-\alpha n^r} n^{1-r} \Big(\frac{1}{\pi \alpha r} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big) E_n(f^{\alpha,r}_{\beta})_{L_1}. \tag{8}$$

In (7) and (8), the quantity $\gamma_{n,p} = \gamma_{n,p}(\alpha, r, \beta)$ is such that $|\gamma_{n,p}| \le (14\pi)^2$.

Proof of Theorem 1. Let $f \in C^{\alpha,r}_{\beta}L_p$, $1 \le p \le \infty$. Then, at every point $x \in \mathbb{R}$ the following integral representation is true:

$$\rho_n(f;x) = f(x) - S_{n-1}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{\alpha,r}_{\beta}(t) P^{(n)}_{\alpha,r,\beta}(x-t))dt, \tag{9}$$

where

$$P^{(n)}_{\alpha,r,\beta}(t) := \sum_{k=n}^{\infty} e^{-\alpha k^r} \cos\Big(kt - \frac{\beta\pi}{2}\Big), \ 0 < r < 1, \ \alpha > 0, \ \beta \in \mathbb{R}. \tag{10}$$

The function $P^{(n)}_{\alpha,r,\beta}(t)$ is orthogonal to any trigonometric polynomial t_{n-1} of degree not greater than $n-1$. Hence, for any polynomial t_{n-1} from we obtain

$$\rho_n(f;x)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\delta_n(t)P^{(n)}_{\alpha,r,\beta}(x-t)dt, \tag{11}$$

where

$$\delta_n(x)=\delta_n(\alpha,r,\beta,n;x):=f^{\alpha,r}_{\beta}(x)-t_{n-1}(x). \tag{12}$$

Further we choose the polynomial t^*_{n-1} of the best approximation of the function $f^{\alpha,r}_{\beta}$ in the space L_p, i.e., such that

$$\|f^{\alpha,r}_{\beta}-t^*_{n-1}\|_p=E_n(f^{\alpha,r}_{\beta})_{L_p},\quad 1\le p\le\infty,$$

to play role of t_{n-1} in (11). Thus, by using the inequality

$$\Big\|\int\limits_{-\pi}^{\pi}K(t-u)\varphi(u)du\Big\|_C\le\|K\|_{p'}\|\varphi\|_p, \tag{13}$$

$$\varphi\in L_p,\quad K\in L_{p'},\quad 1\le p\le\infty,\quad \frac{1}{p}+\frac{1}{p'}=1$$

(see, e.g., [14, p. 43]), we get

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C\le\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_{p'}E_n(f^{\alpha,r}_{\beta})_{L_p}. \tag{14}$$

For arbitrary $\upsilon>0$ and $1\le s\le\infty$ assume

$$\mathcal{I}_s(\upsilon):=\Big\|\frac{1}{\sqrt{t^2+1}}\Big\|_{L_s[0,\upsilon]}, \tag{15}$$

where

$$\|f\|_{L_s[a,b]}=\begin{cases}\Big(\int\limits_a^b|f(t)|^s dt\Big)^{\frac{1}{s}}, & 1\le s<\infty,\\ \operatorname*{ess\,sup}\limits_{t\in[a,b]}|f(t)|, & s=\infty.\end{cases}$$

It follows from the paper [9] for arbitrary $r\in(0,1)$, $\alpha>0$, $\beta\in\mathbb{R}$, $1\le s\le\infty$, $\frac{1}{s}+\frac{1}{s'}=1$, $n\in\mathbb{N}$ and $n\ge n_0(\alpha,r,s')$ the following estimate holds

$$\begin{aligned}\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_s=&e^{-\alpha n^r}n^{\frac{1-r}{s'}}\Bigg(\frac{\|\cos t\|_s}{\pi^{1+\frac{1}{s}}(\alpha r)^{\frac{1}{s'}}}\mathcal{I}_s\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)+\\ &+\delta^{(1)}_{n,s}\Big(\frac{1}{(\alpha r)^{1+\frac{1}{s'}}}\mathcal{I}_s\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r}+\frac{1}{n^{\frac{1-r}{s'}}}\Big)\Bigg),\end{aligned} \tag{16}$$

where the quantity $\delta^{(1)}_{n,s}=\delta^{(1)}_{n,s}(\alpha,r,\beta)$, satisfies the inequality $|\delta^{(1)}_{n,s}|\le(14\pi)^2$.

Substituting $s=p'=\infty$, from 14 and (16) we get (8).

Further, according to [9] for $n \geq n_0(\alpha, r, s')$, $1 < s < \infty$, $\frac{1}{s} + \frac{1}{s'} = 1$, the following equality takes place

$$\mathcal{I}_s\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) = F^{\frac{1}{s}}\Big(\frac{1}{2}, \frac{3-s}{2}; \frac{3}{2}; 1\Big) + \frac{\Theta^{(1)}_{\alpha,r,s',n}}{s-1}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{s-1}, \tag{17}$$

where $|\Theta^{(1)}_{\alpha,r,s',n}| < 2$.

Let now consider the case $1 < p < \infty$.

Formulas (16) and (17) for $s = p'$ and $n \geq n_0(\alpha, r, p)$ imply

$$\frac{1}{\pi}\big\|\mathcal{P}_{\alpha,r,n}\big\|_{p'} = e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma^{(1)}_{n,p}\Big(\frac{1}{p'-1}\frac{(\alpha r)^{\frac{p'-1}{p}}}{n^{(1-r)(p'-1)}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Bigg) =$$

$$= e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma^{(2)}_{n,p}\Big(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{(1-r)(p'-1)}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Bigg), \tag{18}$$

where the quantities $\delta^{(i)}_{n,p} = \delta^{(i)}_{n,p}(\alpha, r, \beta)$, satisfiy the inequality $|\delta^{(i)}_{n,p}| \leq (14\pi)^2$, $i = 1, 2$. Estimate (7) follows from (14) and (18). Theorem 1 is proved. □

It should be noticed, that estimates (7) and (8) are asymptotically best possible on the classes $C^{\alpha,r}_{\beta,p}$, $1 \leq p < \infty$.

If $f \in C^{\alpha,r}_{\beta,p}$, then $\|f^{\alpha,r}_\beta\|_p \leq 1$, and $E_n(f^{\alpha,r}_\beta)_{L_p} \leq 1$, $1 \leq p < \infty$. Considering the least upper bounds of both sides of inequality (7) over the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, we arrive at the inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \leq e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma_{n,p}\Big(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Bigg) E_n(f^{\alpha,r}_\beta)_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{19}$$

Comparing this relation with the estimate of Theorem 4 from [9] (see also [10]), we conclude that inequality (7) on the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, is asymptotically best possible.

In the same way, the asymptotical sharpness of the estimate (8) on the class $C^{\alpha,r}_{\beta,1}$ follows from comparing inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \leq e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\pi\alpha r} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big) E_n(f^{\alpha,r}_\beta)_{L_1} \tag{20}$$

and formula (9) from [8].

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NIERÓWNOŚĆ TYPU LEBESGUE'A DLA SUM FOURIERA NA KLASACH UOGÓLNIONYCH CAŁEK POISSONA

S t r e s z c z e n i e

Dla funkcji ze zbioru uogólnionych całek Poissona $C^{\alpha,r}_{\beta}L_p$, $1 \le p < \infty$, otrzymujemy górne oszacownie dla odchyleń sum Fouriera w jednostajnej metryce w terminach najlepszej aproksymacji uogólnionych pochodnych $f^{\alpha,r}_{\beta}$ funkcji tego typu w metryce przestrzeni L_p. Uzyskane oszacowania są asymptotycznie najlepsze z możliwych.

Słowa kluczowe: nierówności typu Lebesgue'a, sumy Fouriera, uogólnione całki Poissona, najlepsze przybliżenia