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Dedicated to the memory of
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ON CONDITIONINGS OF TENDING TO ZERO SEQUENCES OF RANDOM VECTORS IN BANACH SPACES

Summary

We prove that for a sequence of integrable Banach space valued random vectors (X_n) on non-atomic probability space the following equivalence holds: $\mathbb{E}(X_n|\mathfrak{A}) \to 0$ *a.s.* for any σ-field $\mathfrak{A}$ of events iff $X_n \to 0$ *a.s.* and $\mathbb{E}\,\sup\limits_n \|X_n\| < \infty$.

Thus we extend certain results obtained by Paszkiewicz [3], [4] for real valued random variables.

Keywords and phrases: almost sure (*a.s.*) convergence, conditional expectations, Banach space valued random vectors

1. Introduction

The almost sure convergence of all conditionings for sequences of real valued random variables was studied by Paszkiewicz [3] for positive tending to zero random variables and [4] for arbitrary integrable ones.

A particular version of the main result of [4] can be formulated as follows:

Theorem 1.1. ([4]) *For any non-atomic probability space* $(\Omega, \mathcal{F}, P)$ *and any sequence* (X_n) *of integrable random variables the following conditions are equivalent*

(i) $\mathbb{E}(X_n|\mathfrak{A}) \to 0$ *a.s.* for any $\sigma-$ field $\mathfrak{A} \subset \mathcal{F}$*;*
(ii) $X_n \to 0$ *a.s.* , $\mathbb{E}\sup\limits_n\ |X_n| < \infty$.

Remark. We emphasize that the assumption that $(\Omega, \mathcal{F}, P)$ is non-atomic is essential *(Prop.1.4 in [3])*.

The aim of the present paper is to prove an analogue of Theorem 1.1 for random vectors with values in a Banach space.

2. Preliminaries and notation

Let $(\mathfrak{X}, \|\cdot\|)$ be a real separable Banach space. $\mathfrak{X}^*$ denotes the topological dual of $\mathfrak{X}$ and for $a^* \in \mathfrak{X}^*$ we write $\langle x, a^* \rangle := a^*(x),\ x \in \mathfrak{X}$.

Let us recall that for a given probability space $(\Omega, \mathcal{F}, P)$, an $\mathfrak{X}$-valued random vector is a mapping $X : \Omega \to \mathfrak{X}$ such that for every Borel set $B \subset \mathfrak{X}$, the set $(X \in B) := \{\omega \in \Omega : X(\omega) \in B\}$ belongs to $\mathcal{F}$. A random vector is called <u>simple</u> if it takes on only a finite number of values.
It follows from the separability of $\mathfrak{X}$ that $X : \Omega \to \mathfrak{X}$ is an $\mathfrak{X}$-valued random vector iff for every $a^* \in \mathfrak{X}^*$, $\langle X,\ a^* \rangle : \Omega \to \mathbb{R}$ is a random variable *(see e.g. [2], Prop.1.3)*.

The $\mathfrak{X}$-valued random vector X is <u>integrable</u> (or Bochner integrable) if

$$\mathbb{E}\|X\| := \int_\Omega \|X(\omega)\| P(d\omega) < \infty \ . \tag{2.1}$$

If (2.1) holds, then there exists a sequence (X_m) of simple random vectors such that the sequence $(\|X(\omega) - X_m(\omega)\|)_m$ decreases to zero for all $\omega \in \Omega$ *(see e.g. [2], Lem.1.1)* and hence $\mathbb{E}\|X - X_m\| \downarrow 0$ as $m \to \infty$. Therefore the integral of X (Bochner's integral) can be defined by

$$\mathbb{E}X := \int_\Omega X(\omega) P(d\omega) = \lim_m \int_\Omega X_m(\omega) P(d\omega) := \lim_m \mathbb{E}X_m \ , \tag{2.2}$$

where $\mathbb{E}X_m$ is defined in the standard way. Moreover, the limit in (2.2) does not depend on approximating sequence (X_m) of simple random vectors satisfying $\mathbb{E}\|X - X_m\| \to 0$, and the estimate holds:

$$\|\mathbb{E}X\| \leqslant \mathbb{E}\, \|X\| \ . \tag{2.3}$$

(For details see e.g [2], Sec.1.1.)

For an integrable $\mathfrak{X}$-valued random vector X and a σ-field $\mathfrak{A} \subset \mathcal{F}$ <u>the conditional expectation of X given $\mathfrak{A}$</u>, denoted as $\mathbb{E}_{\mathfrak{A}}X$ or $\mathbb{E}(X|\mathfrak{A})$, is defined as follows.

If X is simple, $X = \sum_{k=1}^{K} a_k \mathbb{1}_{A_k}$, where $a_k \in \mathfrak{X},\ A_k \in \mathcal{F},\ k = i, \ldots,\ K$, one defines

$$\mathbb{E}_{\mathfrak{A}} X = \sum_{k=1}^{K} a_k \mathbb{E}_{\mathfrak{A}} \mathbb{1}_{A_k} \ , \tag{2.4}$$

where $\mathbb{E}_{\mathfrak{A}} \mathbb{1}_{A_k}$ is the standard conditional expectation of indicator random variable, and then (2.4) implies

$$\mathbb{E}\|\mathbb{E}_{\mathfrak{A}} X\| \ \leqslant \ \mathbb{E}\|X\| \ . \tag{2.5}$$

If X is an arbitrary integrable $\mathfrak{X}$-valued random vector, one can define $\mathbb{E}_{\mathfrak{A}} X$ by approximating X by simple random vectors as in (2.2) and using the estimate (2.5). *(See for instance [2], pp 27-28 for more details.)*

Then $\mathbb{E}_{\mathfrak{A}} X$ is a unique (up to a set of probability zero) integrable $\mathfrak{X}$-valued random vector, measurable with respect to $\mathfrak{A}$, such that

$$\int_A \mathbb{E}_{\mathfrak{A}} X dP = \int_A X dP, \ \text{for every } A \in \mathfrak{A} \ .$$

(see e.g. [2], Prop. 1.10).

Moreover,

$$\|\mathbb{E}_{\mathfrak{A}} X\| \ \leqslant \ \mathbb{E}_{\mathfrak{A}} \|X\| \ . \tag{2.6}$$

In the sequel we will refer to the following known property of conditional expectation in $\mathbb{R}$:

Proposition 2.1 (*see e.g. [1], Thm 34.2 (v)*)**.** Let $X_n,\ n \in \mathbb{N},\ X,\ Y$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. If $X_n \to X$ *a.s.* and for all $n,\ |X_n| \leqslant Y$ *a.s.*, where Y is integrable, then for any σ-field $\mathfrak{A} \subset \mathcal{F},\ \mathbb{E}_{\mathfrak{A}} \to \mathbb{E}_{\mathfrak{A}} Y$ *a.s.* .

3. Main Theorem

We can now formulate our main result that extends Theorem 1.1 to the case of Banach space valued random vectors. Thus our Theorem 3.1 below generalizes the main result of [3] and is a partial generalization of the main theorem of [4].

Theorem 3.1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a real separable Banach space. For any non-atomic probability space $(\Omega, \mathcal{F}, P)$ and any sequence of $\mathfrak{X}$-valued random vectors (X_n) the following conditions (i) and (ii) are equivalent* *(i)* X_n *are integrable for all* n *and* $\mathbb{E}(X_n|\mathfrak{A}) \underset{n\to\infty}{\to} 0$ *a.s. for any* σ*-field* $\mathfrak{A} \subset \mathcal{F}$*;*

(ii) (iia) $X_n \to 0$ *a.s. and*

(iib) $\mathbb{E}(\sup_n \| X_n \|) < \infty$.

As an immediate consequence of Theorems 1.1 and 3.1 we obtain:

Corollary 3.2. Under the assumptions of Theorem 1.2 for any sequence (X_n) of integrable $\mathfrak{X}$-valued random vectors, the following conditions are equivalent:

(v) $\mathbb{E}(X_n|\mathfrak{A}) \to 0$ *a.s.* for any σ-field $\mathfrak{A} \subset \mathcal{F}$;

(vv) $\mathbb{E}(\|X_n\| \mid \mathfrak{A}) \to 0$ *a.s.* for any σ-field $\mathfrak{A} \subset \mathcal{F}$.

Remark. A complete generalization of the main result of [4] is under consideration and it can be stated as follows:

Theorem. Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space. For any non-atomic probability space and any sequence (X_n) of integrable $\mathfrak{X}$-valued random vectors the following conditions are equivalent:

(i') for any σ-field $\mathfrak{A} \subset \mathcal{F}$, the sequence $(\mathbb{E}_{\mathfrak{A}} X_n)$ is convergent *a.s* ;

(ii') (X_n) is convergent *a.s.* and $\mathbb{E}(\sup_n \|X_n\|) < \infty$.

It is worth pointing out that in the proof of this theorem, an essentially new argument is needed. The result will appear in a subsequent publication.

4. Proof of Theorem 3.1

Let us note that here the implication "$(i) \Rightarrow (ii)$" is essential and we prove it by reducing to the relevant problem in $\mathbb{R}$ and then using Theorem 1.1.

I. Proof of "$(ii) \Rightarrow (i)$".

By assumption, the real random variables $U_n := \|X_n\| \to 0$ *a.s.* and for all $n \in \mathbb{N}$,

$$0 \leqslant U_n \leqslant Y := \sup_n U_n ,$$

where Y is integrable.

Hence, by Proposition 2.1, for any σ-field $\mathfrak{A} \subset \mathcal{F}$,

$$\mathbb{E}_{\mathfrak{A}} U_n \to 0 \text{ a.s. if } n \to \infty.$$

This and (2.6) imply *(i)*.

II. Proof of "$(i) \Rightarrow (ii)$".

Taking $\mathfrak{A} = \mathcal{F}$, we obtain *(iia)* $X_n \to 0$ *a.s.*

To prove *(iib)* assume on contrary that

$$\mathbb{E}\sup_n \|X_n\| = \infty \ . \tag{4.1}$$

For any fixed m let (Y_{mk}) be a sequence of simple random vectors such that

$$\mathbb{E}\|X_m - Y_{mk}\| \ \to 0 \text{ as } k \to \infty$$

(see Preliminaries).

Hence

$$\forall_m \ \exists_{k_m} \ \forall_{k \geqslant k_m} \ \mathbb{E}\|X_m - Y_{mk}\| < 4^{-m}$$

and denoting $Y_m := Y_{mk_m}$, we have

$$\forall_m \ \mathbb{E}\|X_m - Y_m\| < 4^{-m} \ . \tag{4.2}$$

Since

$$\sup_m \|X_m - Y_m\| \ \leqslant \sum_{m=1}^{\infty} \|X_m - Y_m\| \ ,$$

we obtain by (4.2)

$$\mathbb{E}\sup_m \|X_m - Y_m\| \ \leqslant \sum_{m=1}^{\infty} 4^{-m} = \frac{1}{3} \ . \tag{4.3}$$

From the triangle inequality

$$\|X_m\| \ \leqslant \ \|Y_m\| + \|X_m - Y_m\|$$

we have

$$\sup_m \|X_m\| \ \leqslant \ \sup_m \|Y_m\| + \sup_m \| \ X_m - Y_m\| \ .$$

This and (4.3) imply

$$\mathbb{E}\sup_m \| \ Y_m \ \| \geqslant \ \mathbb{E}\sup_m \| \ X_m \ \| - \frac{1}{3}$$

and hence by (4.1)

$$\mathbb{E}\sup_m \| \ Y_m \ \| = \infty \ . \tag{4.4}$$

Fix m. Since Y_m is simple, it can be written in the form

$$Y_m(\omega) = \sum_{i=1}^{K_m} a_{mi} \mathbb{1}_{A_{mi}}(\omega) \tag{4.5}$$

where $a_{mi} \in \mathfrak{X}$ and $(A_{mi}, i = 1, \ldots, K_m)$ are pairwise disjoint events such that

$$\Omega = A_{m1} \cup \ldots \cup A_{mK_m}, \ P(A_{mi}) > 0 \text{ for all } i \ .$$

Hence for fixed $k \in \{1, \ldots, K_m\}$,

$$\text{for } \omega \in A_{mk}, \ \|Y_m(\omega)\| \ = \ \|a_{mk}\| \ = \sum_{i=1}^{K_m} \ \|a_{mi}\| \ \mathbb{1}_{A_{mi}}(\omega) \ . \tag{4.6}$$

Next, for each $i \in \{1, \ldots, K_m\}$ we take $a^*_{mi} \in \mathfrak{X}^*$ such that

$$\|a^*_{mi}\| \ = 1 \text{ and } \langle a_{mi}, \ a^*_{mi} \rangle = \|a_{mi}\| \ , \tag{4.7}$$

where a_{mi} are from (4.5).

Let

$$V_{mk} := \langle Y_m, \ a^*_{mk} \rangle = \sum_{i=1}^{K_m} \ \langle a_{mi}, \ a^*_{mk} \rangle \ \mathbb{1}_{A_{mi}}, \ k = 1, \ldots, K_m \ .$$

Then from (4.6) and (4.7),

$$\text{for } \omega \in A_{mk}, \ \|Y_m(\omega)\| \ = \ \langle a_{mk}, \ a^*_{mk} \rangle \ = \ V_{mk}(\omega) \leqslant \max_{1 \leqslant i \leqslant K_m} |V_{mi}(\omega)|. \tag{4.8}$$

Let (W_n) denote the sequence of random variables

$$V_{11}, \ldots, V_{1K_1}, V_{21}, \ldots, V_{2K_2}, \ldots \ .$$

Hence by (4.8),

$$\sup_m \|Y_m(\omega)\| \ \leqslant \ \sup_m \max_{1 \leqslant i \leqslant K_m} |V_{mi}(\omega)| \ = \ \sup_n |W_n(\omega)|$$

and by (4.4) we obtain

$$\mathbb{E} \sup_n |W_n| = \infty \ . \tag{4.9}$$

Observe that for each $n \in \mathbb{N}$ there exist uniquely determined $m, \ k \in \mathbb{N}$ such that

$$n = K_0 + \ldots + K_{m-1} + k, \ k \in \{1, \ldots, K_m\}, \text{ where } K_0 := 0 \ . \tag{4.10}$$

Then for n, m, k satisfying (4.10) we have for $\omega \in \Omega$

$$(4.11) \qquad |W_n(\omega)| = |V_{mk}(\omega)| = |\langle Y_m(\omega),\ a^*_{km}\rangle| \leqslant \|\ Y_m(\omega)\ \|\ ,$$

(where the inequality holds, because $\|a^*_{km}\| = 1$).

We now show that

$$(4.12) \qquad \|Y_m\| \ \to 0\ a.s. \text{ if } m \to \infty\ .$$

First, from (4.2) we obtain the estimate:

$$P(\|X_m - Y_m\|\ > 2^{-m}) \leqslant 2^m \mathbb{E}\|X_m - Y_m\|\ \leqslant 2^m \cdot 4^{-m} = 2^{-m},\ m \in \mathbb{N}\ .$$

Then the series $\sum\limits_{m=1}^{\infty} P(\|X_m - Y_m\| \leqslant 2^{-m})$ is convergent and by the Borel-Cantelli Lemma we conclude that

$$(4.13) \qquad \|X_m - Y_m\| \to 0\ a.s. \text{ if } m \to \infty\ .$$

From this, (iia) and triangle inequality, the convergence (4.12) follows.
Next, from (4.11) and (4.12) we conclude that for the real valued random variables (W_n),

$$(4.14) \qquad \lim_n W_n(\omega) = 0\ a.s.\ ,$$

(since for n, m in (4.11) we obviously have: $n \to \infty$ iff $m \to \infty$).
Now, by Theorem 1.1 , from (4.9) and (4.14) we deduce that for some σ-field $\mathfrak{A}_0 \subset \mathcal{F}$,

$$(4.15) \qquad \mathbb{E}_{\mathfrak{A}_0} W_n \nrightarrow 0\ a.s. \text{ if } n \to \infty\ .$$

But for n, m, k as in (4.10) we have

$$|\mathbb{E}_{\mathfrak{A}_0} W_n| = |\mathbb{E}_{\mathfrak{A}_0} \langle Y_m, a^*_{km}\rangle| = |\langle \mathbb{E}_{\mathfrak{A}_0} Y_m,\ a^*_{km}\rangle| \leqslant\ \|\mathbb{E}_{\mathfrak{A}_0} Y_m\|\ a.s.\ ,$$

which implies that

$$(4.16) \qquad \mathbb{E}_{\mathfrak{A}_0} Y_m \nrightarrow 0\ a.s. \text{ if } m \to \infty\ .$$

Finally, we show that

$$\mathbb{E}_{\mathfrak{A}_0} X_m \nrightarrow 0 \ a.s. \ . \tag{4.17}$$

Conversely, suppose that

$$\mathbb{E}_{\mathfrak{A}_0} X_m \to 0 \ a.s. \text{ if } m \to \infty \ . \tag{4.18}$$

By triangle inequality, the linearity of $\mathbb{E}_{\mathfrak{A}_0}(\cdot)$ and (2.6) we have $a.s.$

$$\|\mathbb{E}_{\mathfrak{A}_0} Y_m\| \leqslant \|\mathbb{E}_{\mathfrak{A}_0} X_m\| + \|\mathbb{E}_{\mathfrak{A}_0}(Y_m - X_m)\| \leqslant \|\mathbb{E}_{\mathfrak{A}_0} X_m\| + \mathbb{E}_{\mathfrak{A}_0}\|X_m - Y_m\| \ . \tag{4.19}$$

But $\|X_m - Y_m\| \to 0$ $a.s.$ and $\mathbb{E}\sup\limits_m \|X_m - Y_m\| < \infty$ (see (4.13) and (4.3)), hence by Proposition 2.1 we obtain

$$\mathbb{E}_{\mathfrak{A}_0}\|X_m - Y_m\| \to 0 \ a.s. \ .$$

Combining this with (4.18) and (4.19) we deduce that

$$\mathbb{E}_{\mathfrak{A}_0} Y_m \to 0 \ a.s. \ ,$$

contrary to (4.16). Therefore (4.17) holds, which contradicts the assumption (i). Thus (iib) is proved and the proof of Theorem 3.1 is complete.

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O WARUNKOWANIACH DLA ZBIEŻNYCH DO ZERA CIĄGÓW WEKTORÓW LOSOWYCH W PRZESTRZENIACH BANACHA

S t r e s z c z e n i e

Dowodzimy, że dla ciągu całkowalnych wektorów losowych (X_n) o wartościach w przestrzeni Banacha, określonych na bezatomowej przestrzeni probabilistycznej, zachodzi następująca równoważność:
$\mathbb{E}(X_n|\mathfrak{A}) \to 0$ *a.s.* dla dowolnego σ-ciała $\mathfrak{A}$ zdarzeń wtedy i tylko wtedy, gdy $X_n \to 0$ *a.s.* i $\mathbb{E}\sup\limits_n \|X_n\| < \infty$.
Tym samym uogólniamy pewne wyniki Paszkiewicza [3], [4] otrzymane dla rzeczywistych zmiennych losowych.

Słowa kluczowe: zbieżność prawie pewna (a.s.), warunkowe wartości oczekiwane, wektory losowe w przestrzeniach Banacha