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Dedicated to the memory of
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WEAKLY m-CONVEX SETS AND THE SHADOW PROBLEM

Summary

In this paper we study some properties of weakly m-convex sets in n-dimensional Euclidean space. We obtain estimates for different variants of the shadow problem at a fixed point. We discuss unsolved questions related to this problem.

Keywords and phrases: m-convex set, weakly m-convex set, Grassmann manifold, conjugate set, shadow problem, 1-hull of family of sets

1. Introduction

The purpose of this paper is to study different variants of a problem which can be called the shadow problem at a fixed point. We construct an example giving a lower estimate to create a shadow at a point tangent to the sphere S^2 in the space $\mathbb{R}^3$.

Further, under m-dimensional planes we mean m-dimensional affine subspaces of the Euclidean space $\mathbb{R}^n$.

Definition 1.1. We say that the set $E \subset \mathbb{R}^n$ is *m-convex* with respect to the point $x \in \mathbb{R}^n \backslash E$ if there exists an m-dimensional plane L such that $x \in L$ and $L \cap E = \varnothing$.

Definition 1.2. We say that the open set $G \subset \mathbb{R}^n$ is *weakly m-convex* if it is m-convex with respect to each point $x \in \partial G$ belonging to the boundary of the set G. Any set $E \subset \mathbb{R}^n$ is *weakly m-convex* if it can be approximated from outside by the family of open weakly m-convex sets.

It is easy to construct examples of weakly m-convex set which is not m-convex.

Example 1.1. Let $D = \{(x, y) \mid (|x| < 3, 3 < |y| < 9) \vee (3 < |x| < 9, |y| < 3)\}$ be a set consisting of four open squares. This set is weakly 1-convex, but is not an 1-convex set.

Example 1.2. Let $B = \{(x, y) \mid x^2 + y^2 < 1\}$ be an open circle in the plane. We choose three points of a circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and consider a simplex σ with vertices at these points. It is easy to see that a set $E = B \setminus \sigma$ is also weakly 1-convex set, but it is not an 1-convex.

2. Properties of weakly m-convex sets

In this section we study properties of m-convex sets. The next proposition was proved by Yu. Zelinskii in [5].

Proposition 2.1. *If E_1 and E_2 is weakly k-convex and weakly m-convex set respectively, $k \leqslant m$, then a set $E = E_1 \cap E_2$ is weakly k-convex set.*

Let $G(n, m)$ be Grassmann manifold of m-dimensional planes in $\mathbb{R}^n$[2].

Definition 2.1. A set E^* is called *conjugate* to a set E if E^* is a subset of a set consisting of m-dimensional planes in $G(n, m)$ that don't intersect the set E.

Now we prove the following theorem.

Theorem 2.1. *If K is weakly m-convex compact set and a set K^* is connected then for the section of K by arbitrary $(n-m)$-dimensional plane L the set $L \backslash K \cap L$ is connected.*

Proof. As was proved in the proposition 2 [5] the set K^* is an open set, so any two of its points can be connected by a continuous arc in K^*. Suppose that there exists an $(n-m)$-dimensional plane L for which the set $L \backslash K \cap L$ is not connected. Thus the intersection $K \cap L$ is a carrier of some non-zero $(n-m-1)$-dimensional chain z [2].

Let a point x belong to a bounded component of the set $L \backslash K \cap L$. Such points exist because of the compactness of K. From the weak m-convexity of K it follows that an m-dimensional plane l_1 which does not intersect K passes through the point x. Now we take other m-dimensional plane l_2 outside of some sufficiently large ball containing the compact K.

If we compactificate the space $\mathbb{R}^n$ to a sphere S^n by an infinitely remote point then we obtain two m-dimensional chains $w_1 = l_1 \cup (\infty)$ and $w_2 = l_2 \cup (\infty)$ from which the first chain is affected by the chain z and the second is not. On the one hand, these chains can not be translated into one another by homotopy which would not intersect a chain z and therefore a set $K \cap L$.

On the other hand, from the fact that the set K^* is connected follows the existence in K^* of pairs of points y_1, y_2 which define the planes l_1 and l_2 respectively and connected by an arc in K^*. The points of this arc define the homotopy of the plane l_1 in l_2 that has no common points with the set $K \cap L$. The resulting contradiction completes the proof of our theorem. $\square$

3. The shadow problem

Now we study different variants of the shadow problem in n-dimensional Euclidean space.

For every set $E \subset \mathbb{R}^n$ we can consider the minimal m-convex set containing E and call it *m-convex hull* of a set E.

Introducing the concept of m-convex hull of a set E, we obtain the next problem: to find the criterion that the point $x \in \mathbb{R}^n \setminus E$ belongs to the m-convex hull of the set E. For a case of 1-convex hull of a set that is a union of some set of balls the problem was formulated by G. Khudaiberganov and named *the shadow problem* [1].

The shadow problem. What is the minimum number of mutually disjoint closed or open balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that any straight line passing through the center of the sphere intersects at least one of these balls?

In other words, this problem can be formulated as follows. What is the minimum number of mutually disjoint closed or open balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that the center of the sphere belongs to an 1-convex hull of the family of these balls?

G. Khudaiberganov proved that in the case $n = 2$ two discs are sufficient to create a shadow in the center of a circle. He assumed that for $n > 2$ the minimum number of such balls equals n. Subsequently, professor Yu. Zelinskii [8] proved that in the case $n = 3$ three balls are not enough to create a shadow for the center of the sphere. At the same time the four balls create the shadow. In the general case it is sufficient $n + 1$ balls.

Theorem 3.1. *There exist two closed (open) balls with centers on the unit circle and of radii smaller than one with condition that the center of the circle belongs to an 1-hull of these balls.*

Theorem 3.2. *In order that the center of a sphere S^{n-1} in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$ belongs to an 1-convex hull of a family of mutually disjoint open (closed) balls of radii whose values do not exceed (smaller than) of the radius of the sphere and with centers on the sphere it is necessary and sufficient $(n + 1)$ balls.*

Note that professor Yu. Zelinskii generalized the shadow problem for an arbitrary point inside the sphere.

Problem 3.1. What is the minimum number of mutually disjoint closed or open balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that the interior of the sphere belongs to an 1-convex hull of the family of these balls?

He obtained [9] the solution of this problem in a case $n = 2$.

Theorem 3.3. *In order that an interior of a circle belongs to an 1-convex hull of a family of mutuall disjoint open or closed discs with centers on the circle and of radii smaller than the radius of the circle it is sufficient 3 discs.*

In a case where the point doesn't necessarily belong to some sphere, the following theorem obtained by Yu. Zelinskii is true [7].

Theorem 3.4. *In order that a chosen point in the n-dimensional Euclidean space $\mathbb{R}^n$ for $n \geqslant 2$ belongs to an 1-hull of a family of open (closed) balls that do not contain this point and do not intersect pairwise it is necessary and sufficient n balls.*

Note that where balls are of the same radius we have the next result[10].

Theorem 3.5. *Any set consisting of three balls of the same radius which do not intersect pairwise forms an 1-convex set in the three-dimensional Euclidean space $\mathbb{R}^3$.*

Now we consider a set consisting of three balls in the space $\mathbb{R}^n$. The following statement is true.

Theorem 3.6. *For an arbitrary point of the space $\mathbb{R}^n \setminus \bigcup_{i=1}^{3} B_i$, where B_1, B_2, B_3 are three balls of the same radius that do not intersect pairwise and do not pass through this point, there exists an $(n-2)$-dimensional plane containing this point and does not intersect any of the balls.*

Proof. Let B_1, B_2, B_3 be three balls of the same radius that do not intersect pairwise and do not pass through some point $x \in \mathbb{R}^n$. Let us construct a three-dimensional plane L passing through three centers of the balls and a point x. The intersections of the selected balls with the plane L are three three-dimensional balls B_1', B_2', B_3'. Then according to Theorem 3.5, in the plane L there exists a straight line l which does not intersect any of these balls.

Now we consider the orthogonal complement L_1 of a plane L in the space $\mathbb{R}^n$. This is an $(n-3)$-dimensional plane. Obviously, the Cartesian product $l \times L_1$ is an $(n-2)$-dimensional plane passing through the point x and does not intersect any of the balls B_1, B_2, B_3. The proof is completed. $\square$

Definition 3.1. We say that a family of sets $\Im = \{F_\alpha\}$ creates a shadow tangent to the manifold M at the point $x \in M$ if every straight line tangent to the manifold M

at the point $x \in M \setminus \bigcup\limits_{\alpha} F_\alpha$ has a non-empty intersection at least with one of the sets F_α belonging to the family $\Im$.

Now we formulate the shadow problem for points of the sphere S^{n-1} which don't belong to the union of the balls with respect to the straight lines tangent to the sphere.

Problem 3.2. What is the minimum number of mutually disjoint closed or open balls $\{B_i\}$ in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere which provide a shadow tangent to the sphere S^{n-1} at each point $x \in S^{n-1} \setminus \bigcup\limits_{i} B_i$?

Lemma 3.1. *We consider an equilateral triangle in the Euclidean plane $\mathbb{R}^2$. If we choose three circles B_i, $i = 1, 2, 3$, with centers at the vertices of this triangle and of a radius equals to half of the height of the triangle then every straight line passing through an arbitrary point $x \in (\bigcup\limits_{i=1}^{3} B_i)^* \setminus \bigcup\limits_{i=1}^{3} B_i$, where $(\bigcup\limits_{i=1}^{3} B_i)^*$ is a convex hull of the set $\bigcup\limits_{i=1}^{3} B_i$, intersects at least one of the selected circles.*

Proof. Without loss of generality we take an unit circle with the center at the origin and consider an equilateral triangle with vertices in points $(0, 1)$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ inscribed in the circle. Now we take discs B_1, B_2, B_3 of a radius $\frac{3}{4}$ in each vertex of the triangle. We note that the circumscribed circle of this triangle lies in a convex hull of these three circles.

It is easy to see that any straight line passing through a point $x \in (\bigcup\limits_{i=1}^{3} B_i)^* \setminus \bigcup\limits_{i=1}^{3} B_i$, where $(\bigcup\limits_{i=1}^{3} B_i)^*$ is a convex hull of the set $\bigcup\limits_{i=1}^{3} B_i$, intersects at least one of the three selected discs. By increasing the radii of the selected discs, we obtain that the lemma is true for three open circles of a fixed radius.

Lemma 3.1 gives an answer on the problem 3.2 in the case $n = 2$. $\square$

This result shows that in a three-dimensional case for an arbitrary point of a sphere it is possible to select three balls touching pairwise and creating a shadow at all points of a curvilinear triangle created on the sphere by these balls. Note that the harmonization of such construction for the whole sphere requires additional considerations. This is shown in the following example.

Example 3.1. There exists a set consisting of 14 open (closed) balls that do not intersect pairwise with centers on a sphere $S^2 \subset \mathbb{R}^3$ that can not provide a shadow tangent to the sphere S^2 at each point $x \in S^2 \setminus \bigcup\limits_{i=1}^{14} B_i$.

Without loss of generality we can assume that the chosen sphere S^2 has center at the origin and its radius equals 1. We take a cube with vertices in points ($x = \pm 1/\sqrt{3}, y = \pm 1/\sqrt{3}, z = \pm 1/\sqrt{3}$) inscribed in this sphere. The length of an edge of the cube is equal to $a = 2/\sqrt{3}$.

Now we choose eight open balls with centers at the vertices of the cube and radius $r = 1/\sqrt{3} \approx 0.577$ which equals to half of the cube's edge. We add to this collection six new open balls with centers at an intersection of the rays going from the origin and passing through the center of the face of the cube with the sphere S^2. Radii of these balls equal $r = \sqrt{2 - 2/\sqrt{3}} - 1/\sqrt{3}$. Each of them touches exactly up to four previously selected balls. This collection of balls of two different radii covers the sphere. As the calculations show, this set of balls is not sufficient to create a shadow tangent to the sphere S^2 at each point $x \in S^2 \setminus \bigcup_{i=1}^{14} B_i$.

Note that the constructed set of balls gives a lower estimate of the required number of balls. The question on an upper estimate remains open.

4. Open problems

Unfortunately, Theorem 3.3 gives the solution of the problem 3.1 only in the case $n = 2$. The question on the solution of this problem in higher dimensions remains open.

Question 4.1. What is the minimum number of mutually disjoint closed or open balls in n-dimensional Euclidean space with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that the interior of the sphere belongs to an 1-convex hull of the family of these balls?

Finally, Lemma 3.1 gives the answer on the Problem 3.2 for $n = 2$. At the same time, Example 3.1 gives a lower estimate in the case $n = 3$. The questions on an upper estimate for $n = 3$ and solution of the problem in the case $n > 3$ are open.

Question 4.2. What is the minimum number of mutually disjoint closed or open balls B_i in the space $\mathbb{R}^n$ ($n > 3$) with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere which provide a shadow tangent to the sphere S^{n-1} at each point $x \in S^{n-1} \setminus \bigcup_i B_i$?

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ZBIORY SŁABO m-WYPUKŁE I PROBLEM CIENIA

S t r e s z c z e n i e

Badamy własności zbiorów słabo m-wypukłych w n-wymiarowej przestrzeni euklidesowej. Uzyskujemy oszacowania dla różnych wariantów problemu cienia w ustalonym punkcie. Analizujemy również kilka nierozwiązanych zagadnień.

Słowa kluczowe: zbiór m-wypukły, zbiór słabo m-wypukły, rozmaitość Grassmanna, zbiór sprzężony, problem cienia, 1-otoczka rodziny zbiorów