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# B U L L E T I N

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*Dedicated to the memory of*
*Professor Yurii B. Zelinskii*

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## ON SYSTEM OF BALLS WITH EQUAL RADII GENERATING SHADOW AT A POINT

**Summary**

Problems related to the determination of the minimal number of balls that generate a shadow at a fixed point in the multi-dimensional Euclidean space $\mathbb{R}^n$ are considered in the present work. Here, the statement "a system of balls generate shadow at a point" means that any line passing through the point intersects at least one ball of the system. The minimal number of pairwise-disjoint balls with equal radii in $\mathbb{R}^n$ which do not contain a fixed point of the space and generate shadow at the point is indicated in the work.

*Keywords and phrases:* convex set, problem of shadow, system of balls, sphere, multi-dimensional real Euclidean space

## 1. Introduction

In 1982 G. Khudaiberganov [1] proposed the problem of shadow.

Let us consider $n$-dimensional real Euclidean space $\mathbb{R}^n$ and an open (closed) ball $B(x_0, r) \subset \mathbb{R}^n$ with radius $r$ and center $x$ as the set of all points of distance less than (less than or equal to) $r$ away from $x$ ([2]). It is also called an *n-dimensional ball*. A set of all points in $\mathbb{R}^n$ of the same distance is a sphere $S^{n-1}$ ([2]).

Let $x$ be a fixed point in the real multi-dimensional Euclidean space $\mathbb{R}^n$. We say that a system of balls $\{B_i : i \in \mathbb{N}\} \subset \mathbb{R}^n$ not containing $x$ *generates a shadow at this point* if any straight line passing through $x$ intersects at least one ball of the system. So, the problem of shadow can be formulated as follows: *To find the minimal* *number of pairwise disjoint, open (closed) balls in* $\mathbb{R}^n$ *centered on a sphere* $S^{n-1}$*, not containing the sphere center, and generating shadow at the sphere center.*

This problem was solved by G. Khudaiberganov for the case $n = 2$: it was proved that two balls are sufficient for a circumference on the plane. For all that, it was also made the assumption that for the case $n > 2$ the minimal number of such balls is exactly equal to $n$.

Twenty years passed and Yu. Zelinskii became interested in this problem. In [3], he and his students proved that three balls are not sufficient for the case $n = 3$, but it is possible to generate a shadow at the center of a sphere with four balls. In their work it is also proved that for the general case the minimal number is $n+1$ balls, so the complete answer to this problem for a collection of closed and open balls was obtained. Thus, G. Khudaiberganov's assumption was wrong. In [3], it is also proposed another method of solving the problem for the case $n = 2$ which gives some numerical estimates.

Since 2015, a group of mathematicians leading by Yu. Zelinskii has been working on a series of problems similar to the problem of shadow and their generalizations in the Institute of Mathematics of the National Academy of Sciences of Ukraine. In [4], [5], [9] one can find the review of problems and their solutions related to the problem of shadow. One of these problems is the following:

**Problem 1.** ([6]) *Let* $x$ *be a fixed point of the space* $\mathbb{R}^n$*,* $n \geq 2$*. What is the minimal number* $m(n)$ *of pairwise disjoint, open (closed) balls with equal radii in* $\mathbb{R}^n$ *not containing* $x$ *and generating shadow at* $x$*?*

It is not difficult to show that the minimal number of the balls in the plane is two. In [6], an example of four pairwise disjoint, open (closed) balls with equal radii in space $\mathbb{R}^3$ not containing a fixed point of the space and generating shadow at this point is constructed. In [7] it is proved that non three of such balls in $\mathbb{R}^3$ generate shadow at a fixed point of space. Thus, Problem 1 is solved for space $\mathbb{R}^3$, and the answer is $m(3) = 4$.

Moreover, in [6] it is proved that there does not exist a system of pairwise disjoint, open (closed) balls with equal radii in space $\mathbb{R}^3$ centered on a fixed sphere, not containing the sphere center, and generating shadow at the sphere center.

In the present work, Problem 1 is solved as $n \geq 3$. The following section holds auxiliary results concerning the problem.

The author expresses gratitude to her teacher Professor Yurii Zelinskii for the setting of interesting problems and maintaining author's interest in mathematics through scientific discussions and advices. This work is dedicated to the memory of Professor Yurii Zelinskii.

## 2. Auxiliary results

The results of this section are formulated as lemmas since they are auxiliary within the scope of this paper.

As it was mentioned in the Introduction, the following lemma is true.

**Lemma 1.** ([7]) *Let $n = 3$; then $m(n) = 4$ for any fixed point $x \in \mathbb{R}^3$.*

In [8] a shadow problem for a system of balls with centers freely placed in $\mathbb{R}^n$ without restrictions on their radii is considered. So, the following lemma gives lower estimate for the number of non-overlapping balls that do not contain a fixed point in the space and generate shadow at this point.

**Lemma 2.** ([8]) *The minimal number of open (closed) non-overlapping balls not containing a fixed point in the space $\mathbb{R}^n$, $n \geq 2$, and generating shadow at this point is equal to $n$.*

The following lemma will be frequently used.

**Lemma 3.** ([9]) *Let two open (closed) non-overlapping balls $\{B_i = B(r_i) : i = 1, 2\} \subset \mathbb{R}^n$ with centers on a sphere $S^{n-1}(r)$ and with radii $r_1$, $r_2$ such that $r > r_1 \geq r_2$ be given. Then every ball homothetic to $B_1$ relative to the sphere center with coefficient of homothety $k_1$ does not intersect every ball homothetic to $B_2$ relative to the sphere center with coefficient of homothety $k_2$, if $k_1 \leq k_2$.*

In [3], the following example of system of $n+1$ balls in $\mathbb{R}^n$ satisfying the conditions of Khudaiberganov's shadow problem is given.

**Example 1.** ([3]) Suppose $a$ is the half-length of the edge of an $n$-dimensional regular simplex (see [2]). Let us consider a system of $n + 1$ open balls $\{B_i : i = 1, \dots n + 1\} \subset \mathbb{R}^n$ with correspondent radii $r_1 = a + \varepsilon$, $r_2 = a - \varepsilon/2$, $r_3 = a - \varepsilon/2^2$, $r_4 = a - \varepsilon/2^3$, …, $r_{n+1} = a - \varepsilon/2^n$, where $\varepsilon$ is sufficiently small. Let us place centers of the balls at the vertexes of a simplex such that the balls touch each other. This simplex is slightly different from the regular one and can be inscribed into a sphere. Thus, the system of open balls generate shadow at the sphere center. Let us consider the closures of the balls $\{\overline{B}_i : i = 1, \dots n+1\}$. If we slightly reduce these closed balls, then new system of closed balls generates shadow at the sphere center by continuity.

Using Example 1, in [10] an example of system of $n+1$ balls in $\mathbb{R}^n$ satisfying the conditions of Problem 1 is built as follows.

Let us fix a point $x \in \mathbb{R}^n$ and let us consider open (closed) balls $\{B_i : i = 1, \dots n+1\}$, of Example 1, placed on the sphere with the sphere center at $x$. Let us apply homothety to each open (closed) ball $B_i$ with respective coefficient of homothety $k_i = r_1/r_i$, $i = 1, \dots n + 1$. Then $k_1 < \dots < k_i < k_{i+1} < \dots < k_{n+1}$ and the obtained system consists of $n+1$ balls with the same radii that are equal to $r_1$. Since $r_1 > \dots > r_i > r_{i+1} > \dots > r_{n+1}$, new balls are pairwise disjoint by Lemma 3, do not contain $x$ and generate shadow at $x$ by the constructions. So, the following lemma is true.

**Lemma 4.** ([10]) *Let $n \geq 2$; then $m(n) \leq n+1$ for any fixed point $x \in \mathbb{R}^n$.*

## 3. Main results

We need the following definitions for this section.

Any $m$-dimensional affine subspace of the space $\mathbb{R}^n$, $m < n$, is called an *$m$-dimensional plane.* An $(n-1)$-dimensional plane is called a *hyperplane.*

**Theorem 1.** *Let $n \geq 3$; then $m(n) > n$ for any fixed point $x \in \mathbb{R}^n$.*

*Proof.* The case as $n = 3$ holds by Lemma 1. Let us prove this theorem for $n > 3$ using the method of mathematical induction. We consider the space $\mathbb{R}^4$ with points $x = (x_1, x_2, x_3, x_4)$. Let us fix a point $x_0 \in \mathbb{R}^4$. By Lemma 2, the number of non-overlapping, open (closed) balls generating shadow at $x$ is not less than four. Suppose there exists a system of four non-overlapping four-dimensional open (closed) balls $\{B_i(r, a^i) : i = \overline{1,4}\}$ with the same radii equal to $r$ which does not contain point $x_0$ and generates shadow at $x_0$ (see Fig. 1).

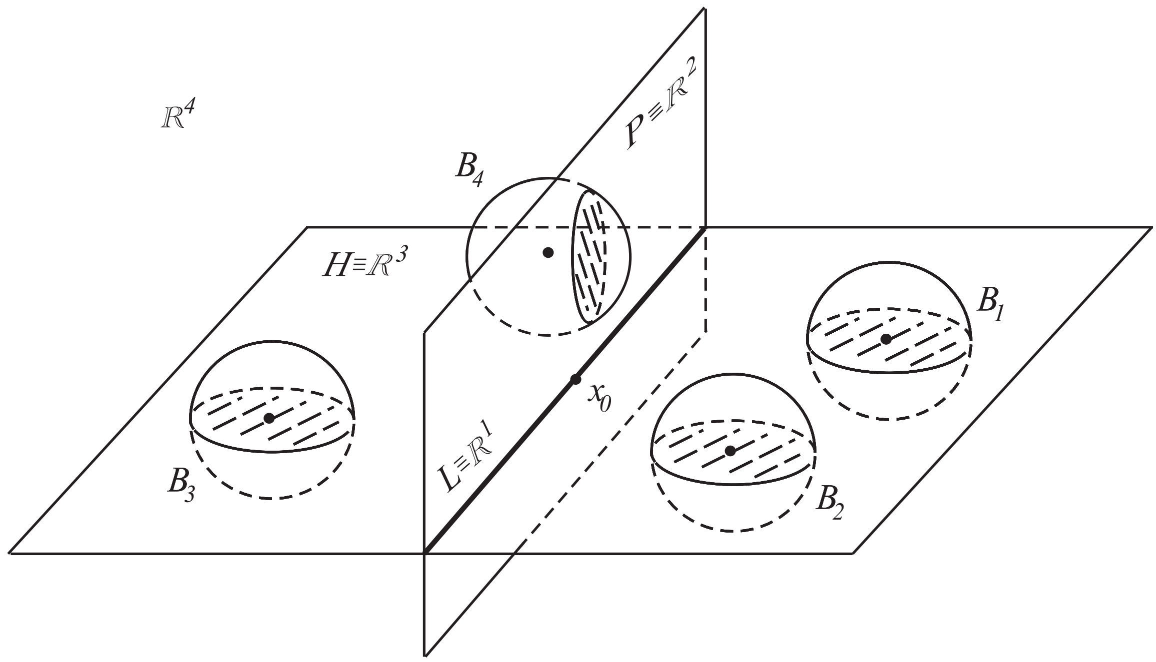


Fig. 1

Then let us draw a three-dimensional plane $H$ (hyperplane) through the point $x$ and the centers of balls $B_1$, $B_2$, $B_3$. Without loss of generality, we can choose a coordinate system such that $H$ is the coordinate hyperplane $x_4 = 0$. Then open (closed) balls $B_i$, $i = 1, 2, 3$, can be described as follows:

$$B_i := \left\{x \in \mathbb{R}^4 : (x_1 - a_1^i)^2 + (x_2 - a_2^i)^2 + (x_3 - a_3^i)^2 + x_4^2 < r^2\right\}$$

$$\left(B_i := \left\{x \in \mathbb{R}^4 : (x_1 - a_1^i)^2 + (x_2 - a_2^i)^2 + (x_3 - a_3^i)^2 + x_4^2 \le r^2\right\}\right),$$

for $i = \overline{1,3}$. The intersection of hyperplane $H$ and balls $B_1$, $B_2$, $B_3$ gives three-dimensional balls with the same radii that are equal to $r$. By Lemma 1 none three non-overlapping open (closed) balls with equal radii in space $\mathbb{R}^3$ generate shadow at point $x_0$. Thus, there exists a straight line $L$ (1-dimensional plain) in hyperplane $H$ passing through $x_0$ and not intersecting any of the three-dimensional balls. Without loss of generality, suppose $L$ coincides with the coordinate axis $x_1$, i.e

$$L := \left\{ x \in \mathbb{R}^4 : \left\{ \begin{array}{c} x_2 = 0, \\ x_3 = 0, \\ x_4 = 0. \end{array} \right. \right\}$$

Then $L \cap B_i = \emptyset$, $i = \overline{1,3}$, i.e

$$B_i : (x_1 - a_1^i)^2 \ge r^2 - a_2^i - a_3^i, \ i = \overline{1,3}$$
$$\left(B_i : (x_1 - a_1^i)^2 > r^2 - a_2^i - a_3^i, \ i = \overline{1,3}\right).$$

We claim that the two-dimensional plane

$$P := \left\{ x \in \mathbb{R}^4 : \left\{ \begin{array}{c} x_2 = 0, \\ x_3 = 0 \end{array} \right. \right\}$$

does not intersect any of the initial four-dimensional balls $B_1$, $B_2$, $B_3$. Indeed,

$$B_i : (x_1 - a_1^i)^2 + x_4^2 \ge (x_1 - a_1^i)^2 \ge r^2 - a_2^i - a_3^i, \ i = \overline{1,3}.$$
$$\left(B_i : (x_1 - a_1^i)^2 + x_4^2 \ge (x_1 - a_1^i)^2 > r^2 - a_2^i - a_3^i, \ i = \overline{1,3}\right).$$

The intersection $P \cap B_4$ is a disk. By Lemma 2, one disk does not generate shadow at $x_0$ in the two-dimensional plane $P$. Thus, in the space $\mathbb{R}^4$ four non-overlapping open (closed) balls with equal radii not containing a point of the space do not generate shadow at the point.

Suppose none $n-1$ non-overlapping, open (closed) balls with equal radii in the space $\mathbb{R}^{n-1}$ not containing a fixed point of the space generate shadow at the point.

Let us consider the space $\mathbb{R}^n$, $n > 4$, and any fixed point $x_0 \in \mathbb{R}^n$. By Lemma 2, the number of non-overlapping, open (closed) balls generating shadow at $x_0$ is not less than $n$. Suppose there exist such $n$ non-overlapping, open (closed) balls $\{B_i(r) : i = \overline{1,n}\} \subset \mathbb{R}^n$ with the same radii equal to $r$ that generate shadow at $x_0$. Then let us draw an $(n-1)$-dimensional plane $H$ through the point $x_0$ and the centers of balls $B_1, \ldots, B_{n-1}$. The intersection of the hyperplane $H$ ($(n-1)$-dimensional plane) and balls $B_1, \ldots, B_{n-1}$ gives $(n-1)$-dimensional balls with the same radii that are equal to $r$. By the assumption these balls do not generate shadow at point $x_0$ in the hyperplane $H$. Thus, there exists a straight line $L$ in $H$ passing through $x_0$ and not intersecting any of the $(n-1)$-dimensional balls. Then, the two-dimensional plane $P$ passing through the straight line $L$ perpendicular to the hyperplane $H$ in $\mathbb{R}^n$ does not intersect any of the initial $n$-dimensional balls $B_1, \ldots, B_{n-1}$. We claim that the ball $B_n$ does not overlap the two-dimensional plane $P$ in the space $\mathbb{R}^n$. Indeed, by Lemma 2, one disk does not generate shadow at $x_0$ in the two-dimensional plane. Thus, in the space $\mathbb{R}^n$ any $n$ non-overlapping, open (closed) balls with equal radii not containing a fixed point of the space do not generate shadow at the point. $\square$

Combining Lemma 4 and Theorem 1, we get the solution of Problem 1, as $n \geq 3$.

**Theorem 2.** *Let $n \geq 3$; then $m(n) = n + 1$ for any fixed point $x \in \mathbb{R}^n$.*

**Remark 1.** Let $n = 2$; then $m(n) = 2$ for any fixed point $x \in \mathbb{R}^2$.

*Proof.* Let us consider a circle with center at a fixed point of the plane and two pairwise disjoint, open (closed) disks in the plane with centers on a circle and radii $r_1$, $r_2$ less than the circle radius generating shadow in the circle center. It is obvious that $r_1 \neq r_2$. Let $r_1 > r_2$ for definiteness. Let us apply homothety to each ball with respective coefficient of homothety $k_i = r_1/r_i$, $i = 1, 2$. Since $k_1 < k_2$, we conclude that obtained balls are pairwise disjoint by Lemma 3, do not contain the sphere center and generate shadow at the sphere center by the constructions. The number of disks is minimal by Lemma 2. $\square$

**Remark 2.** None $n$ pairwise disjoint, open (closed) balls in $\mathbb{R}^n$ centered on a sphere $S^{n-1}$ and not containing the sphere center generate shadow at the sphere center.

*Proof.* This result particular solves Khudaiberganov's shadow problem and is a generalization of well known result for $\mathbb{R}^3$ (see [3], [11]). But, since it is not used in the proof of Theorem 2, we can prove it as follows.

Suppose $n$ pairwise disjoint, open (closed) balls $\{B_i : i = 1, \ldots n\} \subset \mathbb{R}^n$ centered on a sphere $S^{n-1}$ with radii $r_1 \geq \ldots \geq r_i \geq r_{i+1} \geq \ldots \geq r_n$ generate shadow at the sphere center. Let us apply homothety to each ball with respective coefficient of homothety $k_i = r_1/r_i$, $i = 1, \ldots n$. Thus, the obtained system consists of $n$ balls with the same radii that are equal to $r_1$. Since $k_1 \leq \ldots \leq k_i \leq k_{i+1} \leq \ldots \leq k_n$, new balls are pairwise disjoint by Lemma 3, do not contain the sphere center and generate shadow at the sphere center by the constructions. But this contradicts Theorem 2. $\square$

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## O UKŁADZIE KUL Z RÓWNYMI PROMIENIAMI GENERUJĄCYCH CIEŃ W PUNKCIE


S t r e s z c z e n i e

Problemy związane z ustaleniem minimalnej liczby kul generujących cień w ustalonym punkcie w wielowymiarowej przestrzeni euklidesowej $\mathbb{R}^n$ są rozpatrywane przyjmując, że każda prosta przechodząca przez dany punkt przecina jedną z kul układu. Wyznaczona jest minimalna liczba parami rozłącznych kul o równych promieniach w $\mathbb{R}^n$, ktre nie zawierają ustalonego punktu przestrzeni i generują cień w tym punkcie.


*Słowa kluczowe:* zbiór wypukły, problem cienia, układ kul, sfera, wielowymiarowa przestrzeń euklidesowa