https://doi.org/10.26485/0459-6854/2018/68.2/9 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 85–94

Dedicated to the memory of
Professor Yurii B. Zelinskii

Mariia V. Stefanchuk

GENERALIZATION OF THE CONCEPT OF CONVEXITY IN A HYPERCOMPLEX SPACE

Summary

Extremal elements and a h-hull of sets in the n-dimensional hypercomplex space $\mathbb{H}^n$ are investigated. The class of $\mathbb{H}$-quasiconvex sets including strongly hypercomplexly convex sets and closed relatively to intersections is introduced. Some results concerning multivalued functions in the complex space were generalized into the n-dimensional hypercomplex space: there was proved the hypercomplex analogue of the Fenchel-Moreau theorem and some properties of functions that are conjugate to functions $f\colon \mathbb{H}^n \setminus \Theta \longrightarrow \mathbb{H}$.

Keywords and phrases: hypercomplexly convex set, h-hull of a set, h-extremal point, h-extremal ray, $\mathbb{H}$-quasiconvex set, linearly convex function, conjugate function

1. Introduction

The natural analogue of complex analysis is a hypercomplex analysis. Therefore, there is a need to transfer a series of results of a convex analysis known in n-dimensional real and complex spaces, on the n-dimensional hypercomplex space $\mathbb{H}^n$, $n \in N$, which is a direct product of n-copies of the body of quaternions $\mathbb{H}$ [1]. G. Mkrtchyan worked on these problems [2, 3]. He introduced the concepts of hypercomplexly convex, strongly hypercomplexly convex sets and transfered a series of results of linearly convex analysis on hypercomplex space $\mathbb{H}^n$. Yu. Zelinskii [4] and his students (M. Tkachuk, T. Osipchuk, B. Klishchuk) continued to develop this direction.

Let $E \subset \mathbb{H}^n$ be an arbitrary set of the space $\mathbb{H}^n$ containing the origin of coordinates $\Theta = (0, 0, \dots, 0)$. We put $x = (x_1, x_2, \dots, x_n)$, $h = (h_1, h_2, \dots, h_n)$, $\langle x, h\rangle = x_1h_1 + x_2h_2 + \dots + x_nh_n$. The set $E^* = \{h|\langle x, h\rangle \neq 1, \ \forall x \in E\}$ is called the conjugate set to the set E [2].

A hyperplane is called a set $L \subset \mathbb{H}^n$ that satisfies one of the conditions $\langle x, a\rangle = w$, $\langle x - x_0, a\rangle = 0$, where x is an arbitrary point of the set L, x_0 is a fixed vector, w is a fixed scalar with $\mathbb{H}$, and a is a fixed covector. We call the covector a a normal. Accordingly, affine we will call only the functions of the species $l(x) = \langle x, a\rangle + b$, $b \in \mathbb{H}$.

Definition 1 [2]. The set $E \subset \mathbb{H}^n$ is called a hypercomplexly convex if for any point $x_0 \in \mathbb{H}^n \setminus E$ there exists a hyperplane that passes through the point x_0 and does not intersect E.

Definition 2 [2]. The set $E \subset \mathbb{H}^n$ is called a strongly hypercomplexly convex if its arbitrary intersection with the hypercomplex straight line γ is acyclic, that is $\widetilde{H}^i(\gamma \cap E) = 0$, $\forall i \geq 0$, where $\widetilde{H}^i(\gamma \cap E)$ is a consolidated group of Aleksandrov-Cech cohomology sets $\gamma \cap E$ with coefficients in the set of integers.

2. Extremal elements

Let $E \subset \mathbb{H}$ be an arbitrary set. The complement to the union of the unbounded components of the set $\mathbb{H} \setminus E$ is called the h-combination of the points of the set E and is denoted by $[E]$. If E is an arbitrary set in the space $\mathbb{H}^n$, $n > 1$, then we say that the point x belongs to the h-combination of points from E if there exists an intersection of the set E with a hypercomplex straight line γ such that $x \in [E \cap \gamma]$. The set of such points with $\mathbb{H}^n$ is called the h-combination of the points E and denoted $[E]$; the m-multiple h-combination is determined by the induction $[E]^m = [[E]^{m-1}]$ [4].

Definition 3 [2]. The set $\widehat{E} = \cap_\pi \pi^{-1}[\pi(E)]$ is called the h-hull of the set $E \subset \mathbb{H}^n$, where $\pi\colon \mathbb{H}^n \longrightarrow \lambda$ — all possible linear projections of the set on the hypercomplex straight lines, $[\pi(E)]$ is the h-combination of the points of the set $\pi(E)$, and $\pi^{-1}[\pi(E)] = \{x \in \mathbb{H}^n | \ \pi(x) \in \pi(E)\}$ is its complete preimage.

The following theorem [5] asserts that for an arbitrary set of the space $\mathbb{H}^n$ the set of points of its h-hull coincides with the h-combination of the points of this set.

Theorem 1. *If the set $E \subset \mathbb{H}^n$ is an h-hull, then $E = [E]$.*

Proof. Let $x \in [\lambda \cap E]$ for some hypercomplex plane λ. Then, the inclusion $\pi(x) \in [\pi(\lambda \cap E)]$ for all projections π is obviously true, since the restriction of any projection π to each straight line is either homeomorphism or projection into a point. □

Definition 4 [2]. The h-interval with center at the point x of radius r is the intersection of an open ball of radius r with center at the point x with a hypercomplex straight line, which passes through the point x.

Definition 5 [2]. A point $x \in E \subset \mathbb{H}^n$ is called the h-extremal point of the set E if E has no h-intervals containing x.

We extend the Klee's theorem of a convex analysis [6] to a hypercomplex case.

Definition 6 [5]. The h-ray is called a closed unbounded acyclic subset of a hypercomplex straight line with a non-empty boundary.

Definition 7 [5]. The extremal h-ray of the set $E \subset \mathbb{H}^n$ is called the h-ray H belonging to the set E if the set $E \setminus \mathbb{H}$ is hypercomplexly convex and each point of the boundary of the ray H will be an h-extremal point for the set E. (This is equivalent to that no point of the ray H will be internal to the arbitrary h-interval that belongs to the set E and has at least one point outside H).

For the set $E \subset \mathbb{H}^n$ we denote: $\operatorname{hext} E$ is the set of its h-extremal points, $\operatorname{rhext} E$ is the set of h-extremal rays, $\operatorname{hconv} E$ is the h-hull of the set E.

Lemma 1. *Let $E \subset \mathbb{H}^n$ be a closed strongly hypercomplexly convex body ($\operatorname{int} E \neq \emptyset$) with a non-empty strongly hypercomplexly convex boundary ∂E, then E has the form $E = E_1 \times \mathbb{H}^{n-1}$, where E_1 is an acyclic subset of straight line $\mathbb{H}$ with non-empty interior relative to this straight line.*

Proof. Since the boundary ∂E is strongly hypercomplexly convex, then for an arbitrary point $x \in \operatorname{int} E$ there exists a hyperplane that does not intersect ∂E. Therefore, the set E contains a hyperplane. Consequently, by theorem 3 [4], the set E can be depicted in the form of Cartesian product $E = E_1 \times \mathbb{H}^{n-1}$. The set E_1 will be acyclic, because there are intersections E be hypercomplex straight lines that are homeomorphic to E_1. □

Definition 8. An affine subset L is called a tangent to the set E if $L \cap \overline{E} \subset \partial E$, $L \cap \overline{E} \neq \emptyset$.

Lemma 2. *If $E \subset \mathbb{H}^n$ is a strongly hypercomplexly convex closed set and L is its tangent hypercomplex straight line, then* $\operatorname{hext}(E \cap L) = (\operatorname{hext} E) \cap L$.

Proof. Since the inclusion of sets $E \cap L \subset E$ is fair, then by the definition of h-extremal points we have $\operatorname{hext}(E \cap L) \supset (\operatorname{hext} E) \cap L$. Let $x \in \operatorname{hext}(E \cap L)$. Then, inclusion $x \in [K] \setminus K$, where $K \subset E$, can not be performed, because otherwise $K \subset E \cap L$ (since $x \in L$ and L is a hypercomplex straight line, tangent to E). This contradicts the fact that $x \in \operatorname{hext}(E \cap L)$. Consequently, the inverse inclusion of $\operatorname{hext}(E \cap L) \subset (\operatorname{hext} E) \cap L$ is correct and the lemma is proved. □

Remark 1. Analogically, we can prove the equality $\operatorname{rhext}(E \cap L) = (\operatorname{rhext} E) \cap L$ for h-extremal rays.

Theorem 2. *Each closed strongly hypercomplexly convex set $E \subset \mathbb{H}^n$, which does not contain a hypercomplex straight line, will be the h-hull of its h-extremal points and h-extremal rays $E = \operatorname{hconv}(\operatorname{hext} E \cup \operatorname{rhext} E)$.*

Proof. The proof is carried out by induction according to the hypercomplex dimension of the set E. For $\dim_{\mathbb{H}} E = 0$ and $\dim_{\mathbb{H}} E = 1$, the theorem is obvious. Assume that the theorem is valid for all hypercomplex dimensions of the set E, which are less than m $(1 < m \leq n)$. Let us prove it for $\dim_{\mathbb{H}} E = m$.

By the condition of the theorem, the set E does not contain a hypercomplex straight line, therefore it can not coincide neither with its affine hull, nor with the Cartesian product $E_1 \times \mathbb{H}^{n-1}$. Therefore, it follows from lemma 1 that the non-empty boundary ∂E will not be strongly hypercomplexly convex set.

By the definition of a strong hypercomplex convexity, the intersection of the set E with an arbitrary hypercomplex straight line will also be strongly hypercomplexly convex. Let x be an arbitrary point of the set E. If x belongs to a certain tangent straight line L to E, then by the hypothesis of induction we have the inclusion

$$x \in \operatorname{hconv}((\operatorname{hext} E \cap L) \cup \operatorname{rhext}(E \cap L)).$$

If there are points of the set E, through which there is no hypercomplex straight line tangent to E, then there is a point $x \in \operatorname{int} E$.

In this case, we draw a hypercomplex straight line l through the point x. The intersection of $l \cap E$ is a strongly hypercomplexly convex set and does not coincide with l. Therefore, $x \notin [\partial(l \cap E)]$. Now let y be an arbitrary point of the boundary of intersection $\partial(l \cap E)$. Taking into account the strong hypercomplex convexity through the point y, one can draw a straight line T tangent to the set E. By the hypothesis of induction, we obtain $y \in \operatorname{hconv}((\operatorname{hext} E \cap T) \cup \operatorname{rhext}(E \cap T))$. We note that this is fair for every point $y \in \partial(l \cap E)$. Then, taking into account the lemma 2 and the remark 1, we obtain $x \in \operatorname{hconv}(\operatorname{hext} E \cup \operatorname{rhext} E)$. As a result of arbitrariness of choice of the point x we obtain the inclusion $E \subset \operatorname{hconv}(\operatorname{hext} E \cup \operatorname{rhext} E)$. The inverse inclusion is trivial. The theorem is proved. □

3. $\mathbb{H}$-quasiconvex sets

The class of strongly hypercomplexly convex sets is non-closed relatively to the intersection [3]. Therefore, the main axiom of the convexity is not fulfilled: the intersection of any number of convex sets must be convex. We denote the class of sets, which includes strongly hypercomplexly convex sets and is closed relatively to intersections.

Definition 9 [5]. A hypercomplexly convex set $E \subset \mathbb{H}^n$ is called $\mathbb{H}$-quasiconvex set if its intersection with an arbitrary hypercomplex straight line γ does not contain a three-dimensional cocycle, i.e. $\mathbb{H}^3(\gamma \cap E) = 0$.

It is obvious that the class of $\mathbb{H}$-quasiconvex sets includes a strongly hypercomplexly convex domains and compacts.

Let us show the closure of a class of $\mathbb{H}$-quasiconvex sets in the sense that the intersection of an arbitrary family of compact $\mathbb{H}$-quasiconvex sets will be an $\mathbb{H}$-quasiconvex set.

Theorem 3. *The intersection of an arbitrary family of $\mathbb{H}$-quasiconvex compacts will be an $\mathbb{H}$-quasiconvex compact.*

Proof. It is enough to do the proof for two compacts. Let $K_1,\ K_2$ be two arbitrary $\mathbb{H}$-quasiconvex compacts, γ is an arbitrary hypercomplex straight line that intersects the set $K_1 \cap K_2$. We use the exact cohomological sequence of Mayer-Vietoris [7]

$$H^3(\gamma \cap K_1) \oplus H^3(\gamma \cap K_2) \to$$
$$\to H^3(\gamma \cap K_1 \cap K_2) \to H^4(\gamma \cap (K_1 \cup K_2)).$$

Since the compacts K_1 and K_2 are $\mathbb{H}$-quasiconvex, then $H^3(\gamma \cap K_1) = 0$ and $H^3(\gamma \cap K_2) = 0$. Therefore

$$H^3(\gamma \cap K_1) \oplus H^3(\gamma \cap K_2) = 0.$$

On the other hand, a compact intersection

$$\gamma \cap (K_1 \cup K_2) = (\gamma \cap K_1) \cup (\gamma \cap K_2)$$

can not hold the entire hypercomplex straight line γ, which is a four-dimensional real manifold, therefore $H^4(\gamma \cap (K_1 \cup K_2)) = 0$.

From the accuracy of the cohomological sequence it follows that $H^3(\gamma \cap K_1 \cap K_2) = 0$. This is equivalent to the assertion, that the intersection of the set $K_1 \cap K_2$ with an arbitrary hypercomplex straight line does not contain a three-dimensional cocycle. From the previous follows the $\mathbb{H}$-quasiconvexity of the compact $K_1 \cap K_2$. The theorem is proved. □

4. Linearly convex functions

Definition 10 [8]. The function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called multivalued if the image of the point $x \in \mathbb{H}^n$ is a set of $f(x) \in \mathbb{H}$.

The domain of definition of such a function will be denoted by $E_f := \{x \in \mathbb{H}^n : y \in \mathbb{H},\ y = f(x)\}$.

Definition 11. The function $l\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called affine if its graph is a hyperplane.

Definition 12 [8, 9]. A multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called a linearly convex if there exists an affine function $l\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ for an arbitrary pair of points $(x_0, y_0) \in$

$(\mathbb{H}^n \times \mathbb{H}) \setminus \Gamma(f)$ such that $y_0 = l(x_0)$ and $\Gamma(l) \cap \Gamma(f) = \emptyset$ for all $x \in \mathbb{H}^n$, where the graphs of functions l and f, respectively, are denoted by $\Gamma(l)$ and $\Gamma(f)$.

Definition 13. A linearly concave function is called a multivalued function f for which the function $\varphi = \mathbb{H} \setminus f$ is linearly convex.

This means that $\mathbb{H}^{n+1} \setminus \Gamma(f)$ is a graph of a linearly convex function, i.e. through each point $(x_0, y_0) \in \Gamma(f)$ the graph of the affine function passes, which is completely contained in $\Gamma(f)$.

Definition 14 [8, 9]. A multivalued affine function is called a function that is linearly convex and linearly concave simultaneously, and for which there is at least one point $x \in \mathbb{H}^n$, in which each of the sets $(f(x) \cap \mathbb{H})$ and $(\mathbb{H} \setminus f(x))$ is non-empty.

The definition of a linearly convex function can be extended to multivalued functions that take values in an expanded hypercomplex plane $\mathbb{H}^o = \mathbb{H} \cup (\infty)$, compacted by one point.

Here are some examples of linearly convex functions.

Definition 15. A function

$$W_E(y) = \mathbb{H}^o \setminus \cup_{x \in E} \langle x, y \rangle$$

is called the reference function of the set $E \subset \mathbb{H}^n$.

Definition 16. If $E \subset \mathbb{H}^n$ is a linearly convex set, then the function

$$\delta_E(x) = \begin{cases} 0, & \text{if } x \in E, \\ \infty, & \text{if } x \notin E, \end{cases}$$

is called its indicator function.

It is easy to verify that the reference and indicator functions are linearly convex.

Theorem 5. *If f_α, $\alpha \in \mathrm{A}$, is a family of linearly convex functions, where A is an arbitrary set of indices, then the function $f = \cap_{\alpha \in A} f_\alpha$ is linearly convex.*

Proof. We have $\Gamma(f) = \cap_{\alpha \in A} \Gamma(f_\alpha)$. Let us take an arbitrary point

$$(x_0, y_0) \in (\mathbb{H}^n \times \mathbb{H}) \setminus \Gamma(f) = (\mathbb{H}^n \times \mathbb{H}) \setminus \cap_{\alpha \in A} \Gamma(f_\alpha).$$

Then

$$(x_0, y_0) \in (\mathbb{H}^n \times \mathbb{H}) \setminus \Gamma(f_\alpha)$$

for some α, and therefore there is an affine function $l \colon \mathbb{H}^n \longrightarrow \mathbb{H}$ whose graph does not intersect $\Gamma(f_\alpha)$. Therefore, it does not intersect $\Gamma(f)$. Consequently, the function f is linearly convex. The theorem is proved. □

5. Conjugate functions

Definition 17. A function conjugated to f is called a function given by the equality

$$f^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - f(x)). \tag{1}$$

From the definition of conjugate function follows a hypercomplex analogue of Jung-Fenhel's inequality [10]:

$$\langle x, y\rangle \notin f(x) + f^*(y). \tag{2}$$

The correlation (2) can be rewritten in the form

$$\langle x, y\rangle \in \mathbb{H} \setminus (f(x) + f^*(y)),$$

or

$$f(x) \cap (\langle x, y\rangle - f^*(y)) = \emptyset$$

with all $x \in \mathbb{H}^n$, $y \in \mathbb{H}^n$.

We find a function conjugate to a function f^*:

$$f^{**}(x) = (f^*)^*(x) = \mathbb{H}^o \setminus \cup_y(\langle x, y\rangle - f^*(y)).$$

Example 1. Conjugate with a multivalued affine function $f(x) = \langle x, y_0\rangle + f(\Theta)$, where $f(\Theta) \subset \mathbb{H}$ is the set which is the image of the point $\Theta = (0, 0, ..., 0) \in \mathbb{H}^n$, is the function

$$f^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - \langle x, y_0\rangle - f(\Theta)) = \mathbb{H}^o \setminus \cup_x(\langle x, y - y_0\rangle - f(\Theta)) =$$
$$= \begin{cases} \mathbb{H}^o \setminus (-f(\Theta)), & \text{if } y = y_0, \\ \infty, & \text{if } y \neq y_0. \end{cases}$$

Example 2. Let $E \subset \mathbb{H}^n$, $\mathbb{H}^n \setminus E \neq \emptyset$, $f(x) = \delta_E(x)$. Then

$$f^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - \delta_E(x)) = \mathbb{H}^o \setminus \cup_{x \subset E}\langle x, y\rangle,$$

that is, conjugate with the indicator function of its own subset E will be the reference function of this set.

Theorem 6. *For each multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ the inclusion $f \subset f^{**}$ is valid.*

Proof. Let us take an arbitrary pair of points

$$x = (x_1, \ldots, x_n) \in \mathbb{H}^n, \quad y = (y_1, \ldots, y_n) \in \mathbb{H}^n.$$

We obtain from the inequality 2

$$\langle x, y\rangle - f^*(y) \cap f(x) = \emptyset,\ \langle x, y\rangle - f^*(y) \subset \mathbb{H}^o \setminus f(x),$$

i.e.

$$\mathbb{H}^o \setminus (\langle x, y\rangle - f^*(y)) \supset f(x).$$

Taking in the last inclusion the intersection of all $y \in \mathbb{H}^n$, we will obtain such inclusions

$$\cap_y[\mathbb{H}^o \setminus (\langle x, y\rangle - f^*(y))] \supset f(x),$$
$$\mathbb{H}^o \setminus \cup_y(\langle x, y\rangle - f^*(y)) \supset f(x),\ f \subset f^{**}.$$

The theorem is proved. □

Definition 18. A multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called an open (respectively, closed or compact) function when its graph is open (respectively, closed or compact) set in $\mathbb{H}^{n+1}$.

Theorem 7. *Let $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ be a multivalued function. Then the function f^* conjugate to it is linearly convex. If f is open then f^* is closed.*

Proof. The value of the conjugate function can be written as

$$f^*(y) = \cap_x(\mathbb{H}^o \setminus (\langle x, y\rangle - f(x))).$$

For a fixed x the function $y \mapsto \mathbb{H}^o \setminus (\langle x, y\rangle - f(x))$ is a multivalued affine function in y, and therefore it can be presented in the form

$$y \mapsto \langle x, y\rangle + [\mathbb{H}^0 \setminus (-f(x))]. \quad (3)$$

The function f^* is the intersection of linearly convex functions of the form (3), and hence by the Theorem 5 f^* is a linearly convex function. Moreover, if f is open, then each of the functions (3) is closed, and therefore f^* is also closed. The theorem is proved. □

The following theorem is a hypercomplex analogue of the Fenhel-Moro theorem.

Theorem 8. *Let the multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ be such that $\mathbb{H} \setminus f(x) \neq \emptyset$ for all $x \in \mathbb{H}^n$. Then $f^{**} = f$ if and only if when f is linearly convex.*

Proof. We shall show that the equality $f^{**} = f$ is equivalent to the linear convexity of the function f.

If $f^{**} = f$, then, according to the Theorem 7, a function conjugate to an arbitrary function will be linearly convex. If $f(\mathbb{H}^n) \equiv \infty$, then the equality $f^{**} = f$ is obtained from formulas 1 and 2. We have $f^*(y) = \mathbb{H}$ for all $y \in \mathbb{H}^{n*}$ and $f^{**} = \infty$. Since $f \subset f^{**}$ by Theorem 6, it suffices to show that the inverse inclusion $f \supseteq f^{**}$ is valid for a linearly convex function.

Let there be inequality $f(x_0) \neq f^{**}(x_0)$ at some point x_0. Then there is an affine function $l(x) = \langle x, y_0\rangle + \alpha$, such that $\Gamma(l) \cap \Gamma(f) = \emptyset$ and $w_0 = \langle x_0, y_0\rangle + \alpha$, where $w_0 \in f^{**}(x_0) \setminus f(x_0)$. Then

$$f^*(y_0) = \mathbb{H}^o \setminus \cup_x(\langle x, y_0\rangle - f(x)) = \cap_x[\mathbb{H}^o \setminus (\langle x, y_0\rangle - f(x))] \supsetneq (-\alpha),$$

because $[\langle x, y_0\rangle - f(x)] \neq -\alpha$ for all $x \in \mathbb{H}^n$. For the function f^{**} valid is an inclusion

$$f^{**}(x_0) = \cap_y[\mathbb{H}^o \setminus (\langle x_0, y\rangle - f^*(y))] \subset$$

$$\subset \mathbb{H}^o \setminus (\langle x_0, y_0\rangle - f^*(y_0)) \subset \mathbb{H}^o \setminus (\langle x_0, y_0\rangle + \alpha) = \mathbb{H}^o \setminus w_0.$$

Therefore, $w_0 \notin f^{**}(x_0)$, which contradicts the choice of the point $w_0 \in f^{**}(x_0) \setminus f(x_0)$. The theorem is proved. □

Definition 19. Let $f_\alpha\colon \mathbb{H}^n \longrightarrow \mathbb{H}, \alpha \in A$, be multivalued functions. The function $(\cup_\alpha f_\alpha)(x) := \cup_\alpha f_\alpha(x)$ we call the union of functions f_α, and the function $(\cap_\alpha f_\alpha)(x) := \cap_\alpha f_\alpha(x)$ we call their intersection.

For the conjugate functions, there is the theorem of duality.

Theorem 9. *Let $f_\alpha\colon \mathbb{H}^n \longrightarrow \mathbb{H}, \alpha \in \mathrm{A}$, be multivalued functions. Then equality holds*

$$(\cup_\alpha f_\alpha)^* = \cap_\alpha f_\alpha^*.$$

Proof. From expression 1 we obtain for conjugate functions

$$(\cup_\alpha f_\alpha)^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - \cup_\alpha f_\alpha(x)) =$$

$$= \mathbb{H}^o \setminus \cup_x \cup_\alpha (\langle x, y\rangle - f_\alpha(x)) = \mathbb{H}^o \setminus \cup_\alpha \cup_x (\langle x, y\rangle - f_\alpha(x)) =$$

$$= \cap_\alpha(\mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - f_\alpha(x))) = \cap_\alpha f_\alpha^*(y).$$

The theorem is proved. □

References

[1] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex numbers*, Nauka, Moscow, 1973 (in Russian).

[2] G. A. Mkrtchyan, *On hypercomplexly convex sets*, Institute of Mathematics of the AS of USSR, Kiev, 1987 (in Russian).

[3] G. A. Mkrtchyan, *On strong hypercomplex convexity*, Ukr. Math. Journal **2** (1990), 182–187 (in Russian).

[4] Yu. B. Zelinskii, G. A. Mkrtchyan, *On extremal points and hypercomplexly convex domains*, Dokl. The USSR AS **6** (1990), 1299–1302 (in Russian).

[5] M. V. Stefanchuk, *Extremal elements in a hypercomplex space*, Reports of the NAS of Ukraine **4** (2016), 13–19 (in Ukrainian).

[6] K. Leyhtveys, *Convex sets*, Nauka, Moscow, 1985 (in Russian).

[7] E. H. Spanier, *Algebraic topology*, Mir, Moscow, 1971 (in Russian).

[8] M. V. Stefanchuk, M. V. Tkachuk, *Linearly convex and conjugate functions in a hypercomplex space*, Proceedings of the Institute of Mathematics of the NAS of Ukraine **3** (2015), 225–235 (in Ukrainian).

[9] M. V. Stefanchuk, *The properties of conjugate functions in the hypercomplex space*, Proceedings of the International Geometry Center **2** (2017), 39–46 (in Ukrainian).

[10] Yu. B. Zelinskii, *Convexity. Selected chapters*, Institute of Mathematics of the NAS of Ukraine, Kyiv, 2012 (in Russian).

Institute of Mathematics
National Academy of Sciences of Ukraine
Tereshchenkivska st. 3, UA-01004, Kyiv
Ukraine
E-mail: mariast@imath.kiev.ua

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on April 16, 2018.

UOGÓLNIENIE IDEI WYPUKŁOŚCI NA PRZESTRZENIE HIPERZESPOLONE

S t r e s z c z e n i e

Badamy ekstremalne elementy i h-otoczki zbiorów z n-wymiarowej przestrzeni hiperzespolonej $\mathbb{H}^n$. Wprowadzana jest klasa zbiorów $\mathbb{H}$-quasi-wypukłych włączając zbiory silnie hiperzespolenie wypukłe, domknięte w odniesieniu do przecięć. Pewne wyniki dotyczące funkcji wielowartościowych w przestrzeniach zespolonych są uogólnione na przestrzenie hiperzespolone. Dotyczy to twierdzenia Fenchela-Moreau i pewnych własności funkcji sprzężonych do funkcji $f\colon \mathbb{H}^n \setminus \Theta \longrightarrow \mathbb{H}$.

Słowa kluczowe: zbiór hiperzespolenie wypukły, h-otoczka zbioru, punkt h-ekstremalny, zbiór $\mathbb{H}$-guasi-wypukły, funkcja liniowo wypukła, funkcja sprzężona