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DUAL CURVES TO ISOPTICS OF OVALS

Summary

Isoptic curves have been known and studied since the 18th century. Nowadays they have been examined inter alia by Benko, Cieślak, Góźdź, Miernowski and Mozgawa in many papers for example in [1], [2], [3] and [7]. We want to propose a new point of view isoptics. For a given oval we consider a dual curve on Blaschke cylinder and we construct a dual curve for its isoptic. Some properties, for example the loss of convexity is easier to observe on the dual curve than on the given curve. From the analysis of properties of dual curves to isoptics we get a new form of the condition for the convexity of isoptic curves.

Keywords and phrases: isoptic curve, envelope, space of oriented lines, dual curve, Blaschke cylinder

1. Introduction

In this paper, a plane, closed, simple, strictly convex and smooth curve will be called an oval. We introduce a coordinate system with the origin O in the interior of C and denote the support function of C by $p(t)$, where $t \in [0, 2\pi)$. As it was shown in [11] on page 3 and in [8], this support function is differentiable and C can be parameterized by

$$z(t) = p(t)e^{it} + p'(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi), \tag{1}$$

because for a fixed point $z \in C$ we can consider the tangent line l. We can find a point $M \in l$, which is an orthogonal projection of the origin to the line l. If we denote by t the angle between the x-axis and the segment OM, then

$$e^{it} = (\cos t, \sin t), \quad ie^{it} = (-\sin t, \cos t) \quad \text{and} \quad p(t) = |OM|.$$

Note that for any oval C we have $p(t) + p''(t) > 0$ for $t \in [0, 2\pi)$ and the expression $R(t) = p(t) + p''(t)$ is the radius of curvature of C.

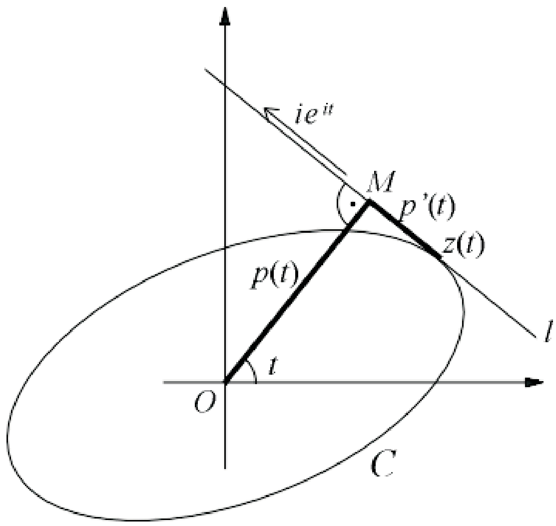

Fig. 1. A parametrization of a convex curve with a support function.

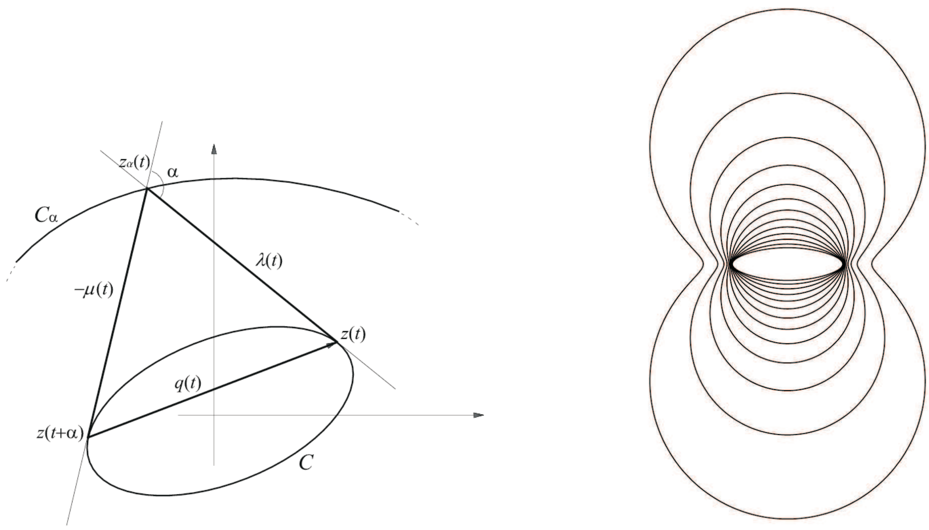

Fig. 2. Construction of an isoptic C_α of the curve C and isoptics of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

For α fixed in the interval $(0,\pi)$ the isoptic C_α of C is a set of points from which the oval C is seen under the angle $\pi-\alpha$. The equation of C_α is

$$z_\alpha(t)=p(t)e^{it}+\{-p(t)\cot\alpha+\frac{1}{\sin\alpha}p(t+\alpha)\}ie^{it},\ t\in[0,2\pi), \tag{2}$$

where $p(t)$ denotes the support function of C (see [2]). Let us notice that

$$z_\alpha(t)=x_\alpha(t)+iy_\alpha(t), \tag{3}$$

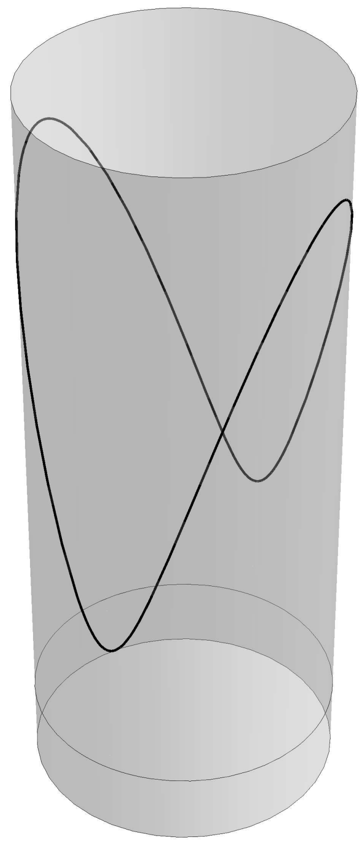

Fig. 3. The dual curve of the ellipse $\frac{x^2}{4^2} + \frac{y^2}{1} = 1$.

where

$$x_\alpha(t) = \frac{1}{\sin\alpha}(p(t)\sin(t+\alpha) - p(t+\alpha)\sin t),$$
$$y_\alpha(t) = \frac{1}{\sin\alpha}(-p(t)\cos(t+\alpha) + p(t+\alpha)\cos t). \tag{4}$$

It is known that the tangent vector at $z_\alpha(t)$ to the isoptic C_α is

$$z'_\alpha(t) = -\lambda(t)e^{it} + \rho(t)ie^{it}, \tag{5}$$

hence

$$x'_\alpha(t) = -\lambda\cos t - \rho\sin t,$$
$$y'_\alpha(t) = -\lambda\sin t + \rho\cos t. \tag{6}$$

Let us notice that $|z'_\alpha(t)| = \sqrt{(x'_\alpha(t))^2 + (y'_\alpha(t))^2} = \frac{|q(t)|}{\sin\alpha}$, where the vector $q(t) = z(t) - z(t+\alpha)$ is important and very convenient in studies of isoptics.

2. The dual curve on the cylinder

The space of oriented lines can be visualized as the Blaschke cylinder $\Gamma = \mathbb{S}^1 \times \mathbb{R}$ as an angle t and the value of the function $p(t)$ defines the line l of equation $\langle OM | e^{it} \rangle = p(t)$ oriented by ie^{it} (see fig. 1). Since a plane curve is the envelope of its support lines, we can associate to each such line a single point on the Blaschke cylinder Γ. This association can be explained in terms of duality of projection geometry (see [4], pp. 10–12, 18–19).

Let us notice that if $C \subset \mathbb{R}^2$ is a plane curve, which can be parameterized by a support function, then we can easily construct its dual curve C^* on the cylinder Γ. Since C can be parameterized by

$$z(t) = p(t)e^{it} + p'(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi),$$

where $p(t) > 0$ denotes the distance between the origin and the tangent line to C at $z(t)$, then the equation of this tangent line is

$$x \cos t + y \sin t - p(t) = 0.$$

Hence to a plane curve we associate the curve on the cylinder Γ by the following formula

$$C^* = \{(\cos t, \sin t, p(t)) : \ t \in [0, 2\pi)\} \tag{7}$$

and we call it dual curve to C. For C with reverse orientation we get

$$C^*_- = \{(\cos t, \sin t, -p(t)) : \ t \in [0, 2\pi)\}.$$

Let us notice that in Γ the oriented line l is coded as a pair $(e^{it}, p) \in \mathbb{S}^1 \times \mathbb{R}$, where t is an angle between the first axis of the coordinate system and the normal vector of the line l. The value p is the distance between the line l and the origin multiplied by the orientation of the line l (1 or -1).

Let us recall that the dual to a pencil of parallel lines with the normal vector $[x, y]$ is the intersection of the cylinder Γ and a vertical plane with normal vector $[x, y, 0]$. The dual to a pencil of lines intersecting in a point $A = [x, y] \in \mathbb{R}^2$ is an ellipse obtained as the intersection of the cylinder Γ and a plane passing through the origin in $\mathbb{R}^3$ and with a normal vector $[x, y, -1]$. Particularly the dual to a pencil of lines intersecting in the origin O is a circle $x^2 + y^2 = 1$ lying on the cylinder Γ.

Hence the shape of the dual curve to a given planar curve depends on the choice of the origin in $\mathbb{R}^2$.

3. The dual curve to the isoptic

Now let us try to construct the dual curve to the isoptic C_α of a given curve C. Let us notice that for C_α we do not have a parametrization with the support function. Moreover as we can see in fig. 2 even the convex curve can have nonconvex isoptics.

Let

$$C: \quad z(t) = p(t)e^{it} + p'(t)ie^{it}$$

be a given oval and let

$$C^*: \quad \{(x, y, z) = (\cos t, \sin t, p(t)),\ t \in [0, 2\pi)\}$$

be a dual curve to C on the cylinder $\Gamma = \mathbb{S}^1 \times \mathbb{R}$. Let us fix an angle $\alpha \in (0, \pi)$.

Let us recall that the point $z_\alpha(t)$ on the isoptic C_α of C is the intersection point of two tangents to C one at $z(t)$ and the other at $z(t+\alpha)$. We already have a dual curve to C so we have dual points to all tangent lines to C. In particular, we have $z^*(t) = (\cos t, \sin t, p(t))$ and $z^*(t+\alpha) = (\cos(t+\alpha), \sin(t+\alpha), p(t+\alpha))$. For a fixed $t \in [0, 2\pi)$ we construct a plane $\pi(t)$ which passes through the origin $O = (0, 0, 0)$ of $\mathbb{R}^3$ and points $z^*(t) = (\cos t, \sin t, p(t))$ and $z^*(t+\alpha) = (\cos(t+\alpha), \sin(t+\alpha), p(t+\alpha))$ on C^*. The intersection of the plane $\pi(t)$ and the cylinder Γ gives an ellipse. Points of this ellipse on the cylinder, that is in the space of oriented lines, correspond with a pencil of lines passing through a point $z_\alpha(t)$ on the isoptic C_α of C. One of these lines is a tangent to the isoptic C_α at $z_\alpha(t)$ and it corresponds to a point $z^*_\alpha(t)$ on the dual curve to C_α. We want to find it.

The normal vector to the plane $\pi(t)$ is

$$N(t, \alpha) = (x_\alpha(t), y_\alpha(t), -1).$$

Let us write the equation of the oriented tangent to isoptic C_α at $z_\alpha(t)$. This line contains $z_\alpha(t)$ and its direction vector is $z'_\alpha(t)$. Hence

$$l = \begin{cases} x = x_\alpha + s \cdot x'_\alpha, \\ y = y_\alpha + s \cdot y'_\alpha, \end{cases} \tag{8}$$

where $s \in \mathbb{R}$ what we can write as

$$y'_\alpha \cdot x - x'_\alpha \cdot y + y_\alpha \cdot x'_\alpha - x_\alpha \cdot y'_\alpha = 0$$

or equivalently

$$\frac{y'_\alpha}{|z'_\alpha|} \cdot x - \frac{x'_\alpha}{|z'_\alpha|} \cdot y - \frac{x_\alpha \cdot y'_\alpha - y_\alpha \cdot x'_\alpha}{|z'_\alpha|} = 0.$$

Let us notice that if $\frac{z'_\alpha(t)}{|z'_\alpha(t)|} = ie^{i\tau}$ for some $\tau \in [0, 2\pi)$ then

$$e^{i\tau} = \left(\frac{y'_\alpha}{|z'_\alpha|}, -\frac{x'_\alpha}{|z'_\alpha|} \right).$$

Hence the dual curve to isoptic C_α is

$$C^*_\alpha = \left(\frac{y'_\alpha(t)}{|z'_\alpha(t)|}, \frac{-x'_\alpha(t)}{|z'_\alpha(t)|}, \frac{[z_\alpha(t), z'_\alpha(t)]}{|z'_\alpha(t)|} \right),\ t \in [0, 2\pi),$$

where $[a + ib, c + id] = ad - bc$.

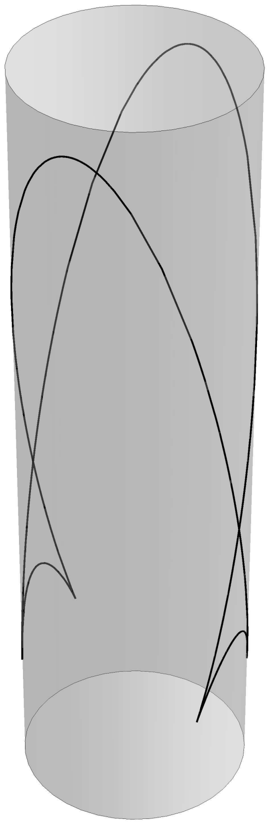

Fig. 4. The dual curve to the isoptic $C_{\frac{3}{4}\pi}$ of the ellipse $\frac{x^2}{4^2} + \frac{y^2}{1} = 1$.

Remark 3.1. We would like to find some kind of generalized support function of isoptics. Let $\tau : [0, 2\pi) \to [0, 2\pi)$ be such function that

$$\frac{y'_\alpha(t)}{|z'_\alpha(t)|} = \cos\tau(t) \quad \text{and} \quad \frac{-x'_\alpha(t)}{|z'_\alpha(t)|} = \sin\tau(t)$$

and $C^*_\alpha = (\cos\tau, \sin\tau, g(\tau))$. Note that for nonconvex isoptic the function $\tau(t)$ is not an injection. Hence the mapping $g(\tau)$ is not a function, since it can have a few different values for the same argument τ (see fig. 4), so it is a kind of multi-valued function.

4. The condition for convexity of isoptics

In [2] the authors gave the following formula for the curvature of the isoptic C_α

$$k_\alpha(t) = \frac{1}{|z'_\alpha(t)|^3}\left[z'_\alpha(t), z''_\alpha(t)\right] = \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)|^2 - [q(t), q'(t)]\right), \tag{9}$$

and in [7] one can find some geometric conditions for convexity of this family of curves. Now we want to propose the new approach to examining of convexity of isoptics.

Theorem 4.1. *Let C be a given oval and let C_α be its α-isoptic for α fixed in $(0, \pi)$. Let*

$$N(t, \alpha) = (x_\alpha, y_\alpha, -1).$$

The isoptic C_α is convex if $\det(N, N', N'') \neq 0$ for all $t \in [0, 2\pi)$, where prime denotes differentiation over t.

Proof. Let us notice that

$$N'(t, \alpha) = \frac{\partial N(t, \alpha)}{\partial t} = (x'_\alpha, y'_\alpha, 0) = (z'_\alpha, 0)$$

and

$$N''(t, \alpha) = \frac{\partial^2 N(t, \alpha)}{\partial t^2} = (x''_\alpha, y''_\alpha, 0) = (z''_\alpha, 0)$$

Hence

$$\det(N, N', N'') = \begin{vmatrix} x_\alpha & y_\alpha & -1 \\ x'_\alpha & y'_\alpha & 0 \\ x''_\alpha & y''_\alpha & 0 \end{vmatrix} = -1 \begin{vmatrix} x'_\alpha & y'_\alpha \\ x''_\alpha & y''_\alpha \end{vmatrix} = -[z'_\alpha, z''_\alpha].$$

From the formula for curvature of the parameterized curve we get

$$k_\alpha = \frac{|[z'_\alpha, z''_\alpha]|}{|z'_\alpha|^3} = \frac{|\det(N, N', N'')|}{|z'_\alpha|^3}.$$

Hence, if C is an oval and $\alpha \in (0, \pi)$, then the curvature $k_\alpha(t) = 0$ if and only if $\det(N, N', N'') = 0$. □

5. Invariants for dual curves to isoptics

For isoptics on the plane there are some geometrical objects as segments of tangents λ and $-\mu$ and the vector q (see fig. 2), which are very useful to describe properties of isoptics. Now we are looking for geometrically indicated segments on the cylinder Γ which correspond to those planar segments.

Corollary 5.1. *Let us fix $t \in [0, 2\pi)$. For the segment of tangent of the length λ, used in construction of α-isoptic of C, there is corresponding vertical segment on the cylinder Γ of length $d_\lambda = \lambda \sin \alpha$ with one end at $z^*(t + \alpha)$.*

Similarly for the segment of tangent of the length $-\mu$ there is a corresponding vertical segment on the cylinder Γ of length $d_\mu = -\mu \sin \alpha$ with one end at $z^*(t)$.

Proof. Let us consider fig. 5. Lines l_1 and l_2 are tangents to the curve C at $z(t)$ and $z(t + \alpha)$, respectively. In the dual space we find ellipses L_1^* and L_2^* corresponding to the pencils of lines intersecting at $z(t)$ and $z(t + \alpha)$. In $\mathbb{R}^2$ by l_3 we denote a line passing through the point $z(t)$ parallel to l_2. Let us notice that the distance between l_2 and l_3 is equal to $d_\lambda = \lambda \sin \alpha$. In the dual space l_2^* and l_3^* are the endpoints of

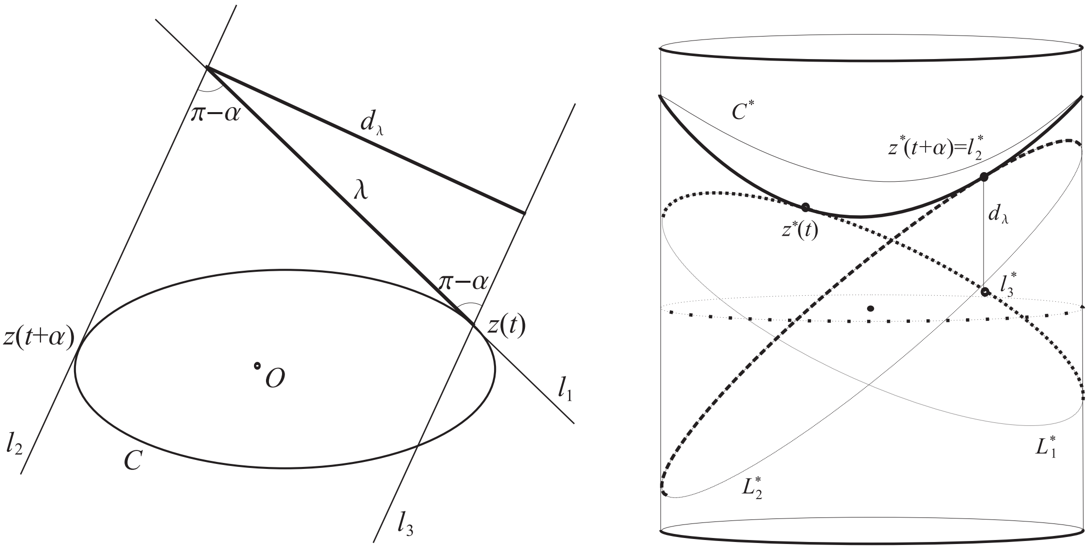

Fig. 5. Invariant for the segment λ.

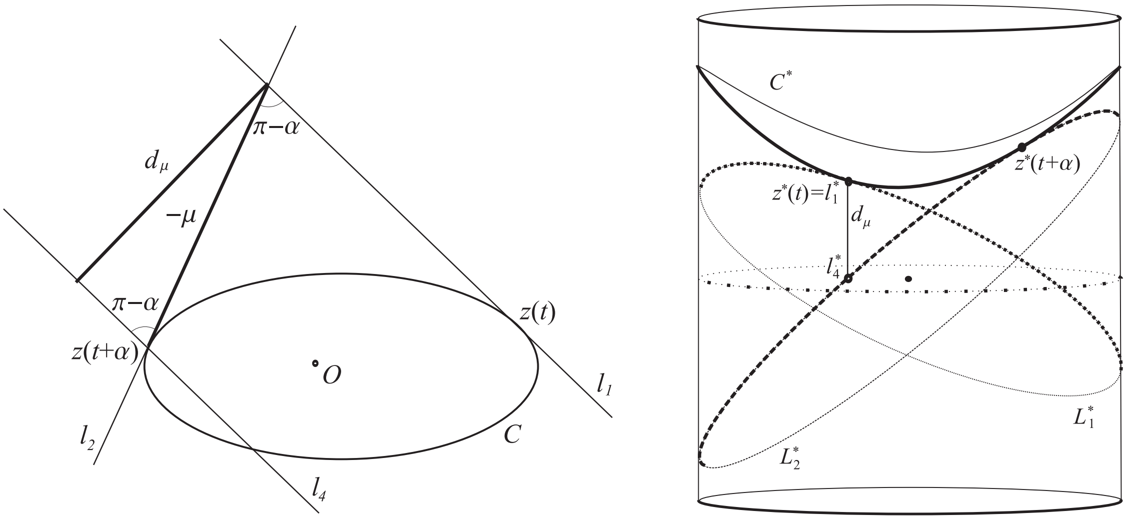

Fig. 6. Invariant for the segment $-\mu$.

a vertical segment of length d_λ. Let us notice that the value d_λ does not depend on the choice of the coordinate system.

In fig. 6 we can see the similar construction for the segment $-\mu$. Let l_4 be the line passing through the point $z(t+\alpha)$ and parallel to l_1. Then the distance between l_1 and l_4 is equal to $d_\mu = -\mu \sin\alpha$. In the dual space l_1^* and l_4^* are the endpoints of a vertical segment of length d_μ.

Additionally, let us notice that the area of the ,,triangle" on Γ with vertices l_1^*, l_2^* and l_3^* is equal to $\frac{1}{2}\alpha d_\lambda$. Similarly, the area of the ,,triangle" on Γ with vertices l_1^*, l_2^* and l_4^* is equal to $\frac{1}{2}\alpha d_\mu$. □

Corollary 5.2. *Let us fix $t \in [0, 2\pi)$. There is a point $-p^*$ on the cylinder Γ which correspond to the oriented line containing the vector $q(t)$, used in construction of*

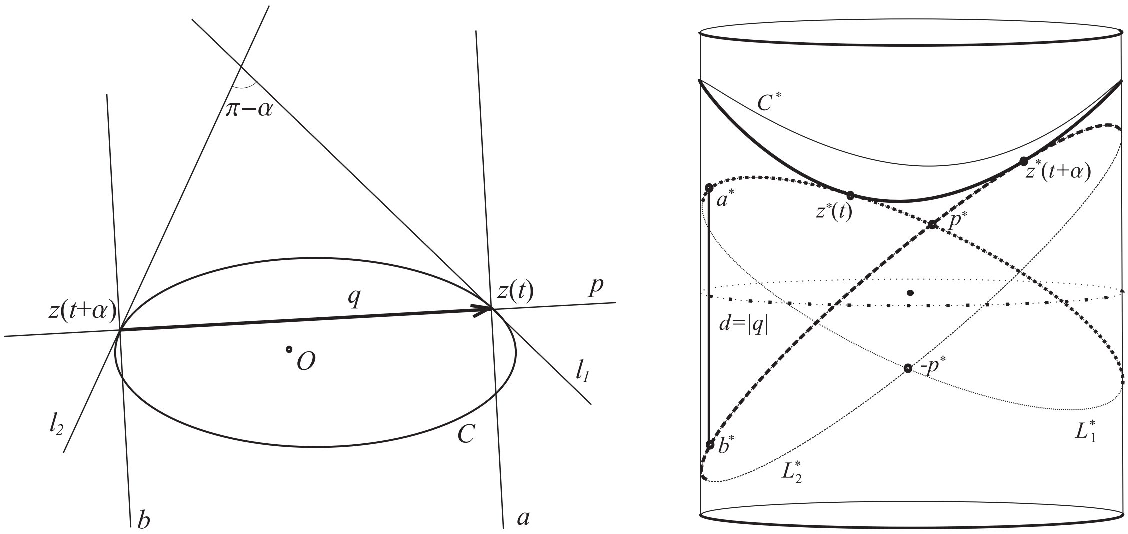

Fig. 7. Invariant for the vector q.

an α-isoptic of C. There is also a geometrically indicated vertical segment of length $d = |q(t)|$ on Γ.

Proof. Let us consider fig. 7. Lines l_1 and l_2 are tangents to the curve C at $z(t)$ and $z(t+\alpha)$, respectively. In the dual space we find ellipses L_1^* and L_2^* corresponding to the pencils of lines intersecting at $z(t)$ and $z(t+\alpha)$. The common points of these two ellipses correspond to the line p which include the vector q. The line oriented in the direction of the vector q is represented by $-p^*$. To find geometrically indicated vertical segment of length $|q(t)|$ on Γ we consider points a^* and b^* on ellipses L_1^* and L_2^* corresponding to lines perpendicular to p and passing through $z(t)$ and $z(t+\alpha)$, respectively. We can get dual points a^* and b^* by moving from p^* along ellipses L_1^* and L_2^* to an angle $\frac{\pi}{2}$. The vertical distance between a^* and b^* is equal to $|q(t)|$. We denote it by d. The area of a ,,triangle" on Γ with vertices a^*, b^* and p^* is equal to $\frac{\pi}{4} \cdot |q(t)|$ and it does not depend on the choice of the origin of the coordinate system. $\square$

Corollary 5.3. *Let us fix $t \in [0, 2\pi)$. For the segment of normal line to C at $z(t)$ of the length $\rho(t)$, used in studies of α-isoptic of C, there is corresponding geometrically indicated vertical segment on the cylinder Γ of length $\rho(t)$ with one end at $z^*(t)$.*

Proof. Let us consider fig. 8. Lines l_1 and l_2 are tangents to the curve C at $z(t)$ and $z(t+\alpha)$, respectively. Let $l_1^\perp$ and $l_2^\perp$ be perpendicular lines to l_1 and l_2 at $z(t)$ and $z(t+\alpha)$, respectively. Let Ω be the intersection of $l_1^\perp$ and $l_2^\perp$. The value of function $\rho(t)$ is equal to the length of the segment joining points $z(t)$ and Ω. Let us denote by l_1^Ω the line parallel to l_1 which passes though Ω. In the dual space we find ellipses L_1^*, L_2^* and Ω^* corresponding to the pencils of lines intersecting at $z(t)$, $z(t+\alpha)$ and Ω, respectively. The value ρ we get as a vertical segment on the cylinder Γ between

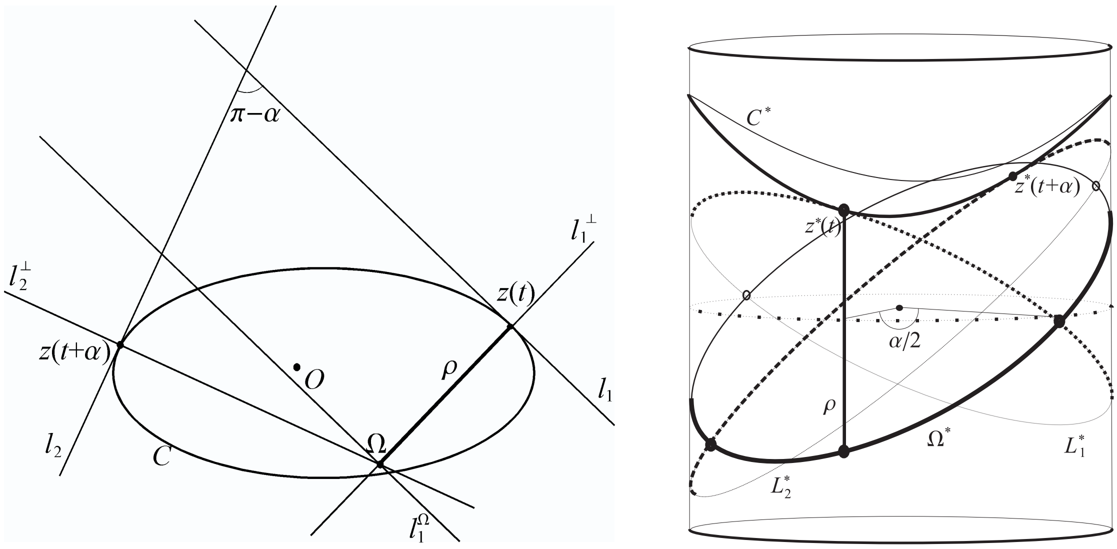

Fig. 8. Invariant for the segment ρ.

L_1^* and Ω^* with one end at $z^*(t)$. □

In the future we plan to construct dual curves for other evolutions of ovals as inner isoptics and evolutoids and examine their properties. We also want to describe limit angles for isoptics of ovals parameterized by a support function $p(t) = a + b\cos kt$, by a function depending on parameters a, b and k. Graphs of functions included in this paper were created in ,,Mathematica", to create other drawings we used ,,Corel".

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KRZYWE DUALNE DO IZOOPTYK OWALI

S t r e s z c z e n i e

Izooptyki, to krzywe, które były znane i rozważane od XVIII wieku. W ostatnim półwieczu badali je między innymi przez Benko, Cieślak, Góźdź, Miernowski i Mozgawa w wielu pracach, na przykład w [1], [2], [3] i [7]. W tej publikacji chcemy zaproponować nowy sposób patrzenia na izooptyki. Dla danego owalu rozważamy jego krzywą dualną na cylindrze Blaschke'go i konstruujemy krzywą dualną do jego izooptyki. Niektóre wasności, na przykład utratę wypukłości, łatwiej jest zaobserwować na krzywej dualnej niż na wyjściowej krzywej. Z analizy własności krzywych dualnych do izooptyk otrzymujemy nową postać warunku na wypukość izooptyk.

Słowa kluczowe: izooptyka, obwiednia, przestrzeń prostych zorientowanych, krzywa dualna, cylinder Blaschke'go