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COMPLETE SUFFICIENT STATISTICS FOR MARKOV CHAINS

Summary

This paper presents a complete sufficient statistic for the class of Markov chains with a finite state S for some natural subspaces $\mathcal{P}_{\mathcal{Z}}$ of the set of all transition probabilities $\mathcal{P}$. This statistic is the random transition count F for Markov chains with a fixed length and a fixed initial state. A summary of the recent results gives a brief exposition of completeness of F with some extra restrictions on the sets $\mathcal{Z}$ and S.

Keywords and phrases: Markov chain, random transition count, complete statistic, minimal sufficient statistic

1. Introduction

Complete sufficent statistics are one of the fundamental concepts in mathematical statistics and play a well known role in estimation theory. In particular these play the essential role in the theory of uniformly most powerful unbiased tests and in minimum variance unbiased estimation. Moreover, we found their application in source coding problems [9]. The concept of completeness of a sufficient statistic comes from Lehmann and Scheffe [3], [4].

This work is an attempt to summarize some of the recent results.

Let $(\mathbf{Y}, \mathcal{B}, \mathbb{M})$ be a statistical space, where $\mathbb{M}$ denotes a parametrized family $\{\mu_\theta : \theta \in \Theta\}$ of probability measures μ_θ on the measurable space $(\mathbf{Y}, \mathcal{B})$.

Definition 1.1. *As usually, we say that a family of distributions* $\mathbb{M} = \{\mu_\theta : \theta \in \Theta\}$ *on* $(\mathbf{Y}, \mathcal{B})$ *is complete if for any* $\mathcal{B}$ *measurable real valued function* g *the condition* $\mathbb{E}_\theta(g) = 0$ for all $\theta \in \Theta$ *implies* $g = 0$ *a.s.* $\mathbb{M}$.

Let $(X, \mathcal{A}, \mathbb{P})$, $\mathbb{P} = \{P_\theta : \theta \in \Theta\}$ be a statistical space and let $F : (\mathbf{X}, \mathcal{A}) \to (\mathbf{Y}, \mathcal{B})$ be a statistic. The statistics F is said to be *complete* if the family of its distributions $\mu_\theta^F = P_\theta(F^{-1}(\cdot))$ on $\mathbf{Y}$, $\theta \in \Theta$ is a complete family.

Fix integer $N \geq 2$, and $S = \{1, \dots, n\}$, $n \geq 2$. We shall use the notation $\mathbf{X}_N = \{(x_1, \dots, x_N) : \ x_i \in S, \ i = 1, 2, \dots, N\}$ for a Markov chain with a finite state space S and with a fixed length N. The evolution law is described by an unknown initial distribution $q = (q_1, \dots, q_n)$, where $q_i = P(x_1 = i)$, and an unknown stochastic matrix $p = (p_{i,j})$. The space of such distributions we denote by $\mathcal{Q}$ and the space of all such matrices by $\mathcal{P}$,

$$\mathcal{P} = \{p = (p_{i,j}) : \forall_{i,j \in S} \quad p_{i,j} \geq 0, \quad \sum_{j=1}^{n} p_{i,j} = 1 \text{ for each } i \in S\}.$$

In other words our family of Markov chains is parametrized by $\mathcal{Q} \times \mathcal{P}$.

Now, recall the definition of the random transition count F,

$$F(\mathbf{x}) = F(x_1, \dots, x_N) = (f_{i,j})_{i,j=1,\dots,n},$$

$$f_{i,j} = \#\{t = 1, \dots, N-1 : \quad x_t = i, \ x_{t+1} = j\} \text{ for } \ i, j \in S.$$

Obviously, F is a basic tool in any statistical investigation. It is well known that for fixed x_1, F is a sufficient statistic. The problems of completeness of the random transition counts still have not been investigated enough. Denny and Wright [1] showed that if the initial state x_1 is fixed then F is a complete sufficient statistic for $\mathcal{P}$, but the proof was delicate. We note that the situation changes when x_1 is not fixed. It turns out that F is not sufficient and the natural statistic $(x_1, F(\mathbf{x}))$ is sufficient but in general is not complete.

Proposition 1.2. *Let $S = \{1, 2\}$, $N = 2$. The statistic $G(x_1, x_2) = (x_1, (f_{i,j}))$ is sufficient, but not complete.*

Proof. Indeed, for a fixed initial distribution (q_1, q_2); $q_1, q_2 \in [0, 1]$, $q_1 + q_2 = 1$ and for any transition probability matrix

$$p = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix} \in \mathcal{P},$$

the statistic G takes the following values with the corresponding probabilities:

$$P(G = G_1) = q_1 \cdot p_{1,1} \ \text{ for } \ G_1 = \left(1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right),$$

$$P(G = G_2) = q_1 \cdot p_{1,2} \ \text{ for } G_2 = \left(1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right),$$

$$P(G = G_3) = q_2 \cdot p_{2,1} \ \text{ for } G_3 = \left(2, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right),$$

$$P(G = G_4) = q_2 \cdot p_{2,2} \quad \text{for } G_4 = \left(2, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Then the expectation $\mathbb{E}_{q,p}(g \circ G)$ vanishes whenever the function g satisfies $g(G_1) = g(G_2) = q_2,\ g(G_3) = g(G_4) = -q_1.$ □

However, for a fixed N, S : there exists a complete sufficient statistic for the set of all stationary chains if and only if $S = \{1, 2\}$. It is mentioned on p. 338 of Denny and Wright [1]. By definition, the parameter space for the set of all stationary Markov chains with state space S is $\mathcal{P}_S = \{(p_{i,j}) :\ (p_{i,j}) \in \mathcal{P},$ there exists at least one stationary initial distribution (p_i) for $(p_{i,j})\}$. The following function is a complete sufficent statistic for $\mathcal{P}_S$:

$$G(x_1, \ldots, x_N) = \begin{cases} (x_1, f_{1,1}, f_{2,2}, 0) & if\ f_{1,2} = f_{2,1}, \\ (0, f_{1,1}, f_{2,2}, f_{1,2} + f_{2,1}) & if\ f_{1,2} \neq f_{2,1}. \end{cases}$$

Next, it is shown in [10] that a complete sufficient statistic for the class of all stationary n-state Markov chains, $n \geq 3$, does not exist.

Theorem 1.3. *If $n \geq 3$ e.g. $S \neq \{1, 2\}$, then $(x_1, (f_{i,j}))$ is minimal sufficient statistic for $\mathcal{P}_S$ which is not complete. Consequently there does not exist a complete sufficient statistic for $\mathcal{P}_S$.*

2. The statistical assumptions

This paper is devoted to study completeness of the random transition count for largest possible class of Markov chains. The crucial concept is that we use stochastic matrices $\mathcal{P}_\mathcal{Z} \subset \mathcal{P}$:

Let $\mathcal{Z} \subset S \times S$ denote a fixed subset satisfying

$$\forall_{i \in S} \quad \exists_{j \in S} \quad (i, j) \notin \mathcal{Z}. \tag{1}$$

We denote by $\mathcal{P}_\mathcal{Z}$ the set of stochastic matrices $p \in \mathcal{P}$, such that

$$\forall_{(i,j) \in \mathcal{Z}} \quad p_{i,j} = 0. \tag{2}$$

Now we assume that the initial state x_1 is fixed (e.g. $q_i = \delta_{x_1}(i),\ i \in S$). Thus probability distribution on the space $\mathbf{X}_{x_1,N} = \{x_1\} \times S^{N-1}$ is given by the formula

$$P_{(p_{i,j})}(\{\mathbf{x}\}) = p_{x_1,x_2} \cdot\ \ldots\ \cdot p_{x_{N-1},x_N} \quad \text{for} \quad \mathbf{x} = (x_1, \ldots, x_N) \in \mathbf{X}_{x_1,N}.$$

Throughout the paper we are dealing with a statistical space of the form

$$(\mathbf{X}_{x_1,N}, \{P_{(p_{i,j})} :\ (p_{i,j}) \in \mathcal{P}_\mathcal{Z}\}).$$

Obviously, the assumption $p \in \mathcal{P}_\mathcal{Z}$ means that the transitions in one step from i to j are forbidden for $(i, j) \in \mathcal{Z}$. The space $\mathcal{P}_\mathcal{Z}$ remains non empty because of (1).
This condition fits well to the characterization of some classical types of Markov chains.

Example 2.1. Assume that $S = \mathbb{Z}$. Then $(x_1, \ldots, x_N)$ is a random walk with stationary transition probability, possibly depending on position and direction, if and only if the matrix p is taken from $\mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(i, i+\epsilon); i \in S,\ \epsilon = \pm 1\}.$$

Example 2.2. Assume now that $S = \{1, \ldots, d\}^k$, $x_t = (y_t^1, \ldots, y_t^k) \in S$ for $t = 1, \ldots, N$, and let the evolution of x_t be given by a matrix $(p_{\mathbf{ij}})$, $\mathbf{i} = (i^1, \ldots, i^k)$, $\mathbf{j} = (j^1, \ldots, j^k) \in S$. Then (x_t) is a Markov chain of order k, with a state space $S_1 = \{1, \ldots, d\}$, if and only if $(p_{\mathbf{ij}}) \in \mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(\mathbf{i}, \mathbf{j}) : (i^2, \ldots, i^k) = (j^1, \ldots, j^{k-1})\}.$$

The random transition count F could be non-complete for some space $\mathcal{P}_{\mathcal{Z}}$, what we may see in an

Example 2.3. Let $S = \{1, 2, 3, 4\},\ N = 5$ and let $x_1 = 3$ be fixed. For the space $\mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(1,3), (2,3), (3,2)(3,4), (4,1)\}$$

the statistic F is not complete.

Proof. For the set $\mathcal{Z}$ any matrix $p \in \mathcal{P}_{\mathcal{Z}}$ can be written as

$$p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1-q \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad q \in [0, 1].$$

The statistic $F(\mathbf{x}) = (f_{i,j})$ takes on the following values with corresponding probabilities

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$P(F(\mathbf{x}) = M_1) = (1-q)q, \qquad P(F(\mathbf{x}) = M_2) = (1-q)q,$$

Then the expectation $\mathbb{E}_p(g \circ F)$ vanishes for any non-zero function g satisfying $g(M_1) = -g(M_2)$. $\square$

Thus it is necessary to put some extra assumptions about the set $\mathcal{Z}$ and S (cf. condition in Section 4).

3. Auxiliary results and the graph theory

For the convenience of the reader we repeat the relevant material from [5], [6], [7] without proofs, thus making our exposition self-contained.

In what follows, we introduce a specific notation concerning "tables" of numbers. The proofs of lemmas are based on some combinatorical tools.

For any matrix f of dimension $n \times n$, let $f' = (f'_{i,j})$ denote a "table" being the matrix f with deleted elements $f_{i,i+1},\ i = 1, \dots, n\ (f_{n,n+1} \equiv f_{n,1})$.
More precisely

(3) $$f'_{i,j} = f_{i,j} \qquad \text{for} \quad j \neq i+1 \pmod{n}$$

and

(4) $$f' = (f'_{i,j})_{i=1,\dots,n;\ j=1,\dots,i,i+2,\dots,n.}$$

Lemma 3.1. *Fix $i', i'' \in S$ and $N \geq 2$. Let $\mathcal{M}_{N-1}^{i',i''}$ denote a set of matrices f of dimension $n \times n$ satisfying*

(i) $\sum_{i,j} f_{i,j} = N - 1$,

(ii) $\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i)$ *for* $i \in S$,

with δ being the Kronecker delta. The function $f \to f'$ defined by (3), (4) *is one-one on the class $\mathcal{M}_{N-1}^{i',i''}$. There exist functions $\phi_i^{i',i''}$ on tables f' satisfying (4), such that*

$$\phi_i^{i',i''}(f') = f_{i,i+1} \qquad \textit{for } i \in S.$$

Lemma 3.2. *Fix $i' \in S$ and $N \geq 2$. Let $\mathcal{M}_{N-1}^{i'}$ denote a set of matrices f of dimension $n \times n$ with integer elements satisfying*

(I) $\sum_{i,j} f_{i,j} = N - 1$,

(II) *there exists $i'' \in S$ such that*

$$\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i) \qquad \textit{for} \quad i \in S.$$

The function $f \to f'$ defined by (3), (4) *is one-one on the class $\mathcal{M}_{N-1}^{i'}$. There exist functions $\phi_i^{i'}$ with integer values, such that*

$$\phi_i^{i'}(f') = f_{i,i+1} \qquad \textit{for } i \in S,$$

for f' satisfying (3) *and* (4).

Corollary 3.3. *The function $f \to f'$ defined by* (3), (4) *is one-one on the class $F[\mathbf{x}]$ of the random transition counts F for the trajectories $\mathbf{x} \in \mathbf{X}$ with a fixed initial state $x_1 = i'$ and a fixed length $N \geq 2$.*

Fix $i', i'' \in S = \{1, \ldots, n\}$. For any matrix f of dimension $n \times n$, let $\bar{f} = (\bar{f}_{i,j})$ denote a "table" being the matrix f with deleted elements $f_{i,i+1}$, $i = 1, \ldots, n$, $i \neq i''$, ($f_{n,n+1}$ is another notation for $f_{n,1}$). More precisely we use a set of indices

$$S_2^{i',i''} = \{(i,j) \in S \times S : \ j \in S \setminus \{i+1\} \quad \text{for} \quad i \in S \setminus \{i'', n\},$$
$$j \in S \quad \text{for} \quad i = i'', \quad j \in S \setminus \{1\} \quad \text{for} \quad i = n \quad \text{in the case} \quad n \neq i''\}$$

and define

$$\bar{f}_{i,j} = f_{i,j} \qquad \text{for } (i,j) \in S_2^{i',i''}, \tag{5}$$

$$\bar{f} = (\bar{f}_{i,j})_{(i,j) \in S_2^{i',i''}}. \tag{6}$$

Lemma 3.4. *Fix $i', i'' \in S$. Let $\mathcal{M}^{i',i''}$ denote a set of matrices $f = (f_{i,j})$ of dimension $n \times n$ satisfying*

$$\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i) \quad \textit{for} \quad i \in S.$$

The function $f \to \bar{f}$ defined by (5), (6) is one-one on the class $\mathcal{M}^{i',i''}$. There exist functions $\psi_i^{i',i''}$ for $i \neq i''$, defined on tables $(f_{i,j})_{(i,j) \in S_2^{i',i''}}$, with non-negative integer values satisfying

$$\psi_i^{i',i''}(\bar{f}) = f_{i,i+1} \qquad \text{for} \quad i \in S \setminus \{i''\} \text{ and any } \ f \in \mathcal{M}^{i',i''}.$$

Corollary 3.5. *The function $f \to \bar{f}$ defined by* (5), (6) *is one-one on the class $\{F(\bar{\mathbf{x}}) : \quad \bar{\mathbf{x}} = (x_1, \ldots, x_t), \ x_1 = i', \ x_t = i'', \ 1 \leq t < N\}$ of all values of random transition counts for the trajectories $\bar{\mathbf{x}}$ with a fixed initial state i' and a fixed final state i'' (and any length).*

We will need the following lemma which was used by Denny and Wright. Assume that integers $n \geq 1, \ j(1) \geq 1, \ldots, j(n) \geq 1, \ q \geq 0$ and real $c > 0$ are fixed. Denote by $\mathcal{U}$ the set of all systems of positive numbers $u = (u_{i,j}), \ 1 \leq i \leq n, \ 1 \leq j \leq j(i)$ such that

$$\sum_{j=1}^{j(i)} u_{i,j} \leq c \qquad \text{for any} \quad 1 \leq i \leq n.$$

Denote by M the set of systems of non-negative integers $m = (m_{i,j})$, $1 \leq i \leq n$, $1 \leq j \leq j(i)$ satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{j(i)} m_{i,j} \leq q.$$

Let $\varphi_i : M \to \{0,1,2,\ldots\}$, $i = 1,\ldots,n$ be any given functions. For each $m \in M$ let us define a function $W_m : \mathcal{U} \to \mathbb{R}$, as follows

$$W_m(u) = \prod_{i=1}^{n} \Big[\prod_{j=1}^{j(i)} u_{i,j}^{m_{i,j}} \cdot \big(c - u_{i,1} - \ldots - u_{i,j(i)}\big)^{\varphi_i((m_{i,j}))} \Big].$$

Lemma 3.6. *The system of functions $W_m : \mathcal{U} \to \mathbb{R}$, indexed by $m \in M$, is linearly independent.*

Proof. Cf. Lemma 2 in [1]. □

The following lemmas from graph theory go back to the work by A. Paszkiewicz.

Definition 3.7. *Let $Z \subset S \times S$ be a fixed set, fix $i', i'' \in S$. We say that an oriented graph (Y,U), $Y \subset S$, with Y being the set of vertices and $U \subset Y \times Y$ being the set of edges, is defined by Z, i', i'' if*

$$Y = \bigcup \{x_1, \ldots, x_N\}, \; U = \bigcup \{(x_1,x_2), \ldots, (x_{N-1}, x_N)\}$$

with the unions taken for all sequences (x_t) satisfying $(x_t, x_{t+1}) \notin Z$, $t = 1,\ldots,N-1$, and $x_1 = i'$, $x_N = i''$.

As usual, we say that (Y_0, V_0) is a cycle if

$$Y_0 = \{y_1, \ldots, y_s\},$$
$$V_0 = \{(y_1,y_2), \ldots, (y_{s-1}, y_s), (y_s, y_1)\}.$$

A graph (Y_1, V_1) is a tree with root y if for any $z \in Y_1$ there exists exactly one path (Y_z, V_z) of the form

$$Y_z = \{z_1 = z, z_2, \ldots, z_s = y\} \subset Y_1,$$
$$V_z = \{(z_1, z_2), \ldots, (z_{s-1}, z_s)\} \subset V_1,$$

($z_1, \ldots, z_s$ are mutually different and $s \geq 1$, with $V_z = \emptyset$ for $s = 1$).

Lemma 3.8. *For any graph (Y,U) defined by Z, i', i'' there exists a tree (Y,W), $W \subset U$, with root i''.*

Let F be the random transition count and let the evolution be given by transition probabilities p from $\mathcal{P}_{\mathcal{Z}}$ with a fixed set $\mathcal{Z}$. In the following lemma we use the notion of a tree to describe some general properties of F.

Let $Y = \{1,\ldots,n\}$ and let a graph (Y,U) be defined by Z, i', i''. The value $F(\mathbf{x}) = (f_{i,j})$ for any trajectory $\mathbf{x}$ and satisfies $f_{i,j} = 0$ for $(i,j) \notin U$. Thus $F(\mathbf{x})$ can be identified with some function $m : U \to \{0,1,\ldots\}$, $m = f|_U$, and obviously

$$\sum_{j \in Y, (i,j) \in U} m_{i,j} + \delta_{i''}(i) = \sum_{j \in Y, (j,i) \in U} m_{j,i} + \delta_{i'}(i) \quad \text{for} \quad i \in Y.$$

Denote by M the space of all such function $m : U \to \{0, 1, 2, \ldots\}$.

Lemma 3.9. *Let (Y, U) be defined by Z, i', i''. For any tree (Y, V) with root i'' and with $V \subset U$, denote by $j(\cdot)$ the uniquely defined function on $Y \setminus \{i''\}$ satisfying $(i, j(i)) \in V$. Then there exist functions Φ_i on $\{m|_{U \setminus V}; m \in M\}$ with non-negative integer values satisfying*

$$m_{i,j(i)} = \Phi_i(m|_{U \setminus V})$$

for any $i \in Y$, $i \neq i''$, $m \in M$.

4. Completeness of the random transition count

In Section 1 we reviewed some of the standard facts of the complete statistic. Now we present some recent results of the completeness of the random transition count for some special classes of transition probabilities.

We know that the random transition count F is complete for Markov chains with a fixed length and a fixed initial state. Moreover, the statistic F is always complete for Markov bridge.

The random walk $\mathbf{x} = (x_1, \ldots, x_N)$ is a *Markov bridge*, if $x_1 = i'$, $x_N = i''$ are fixed. Thus in canonical representation

$$P_{(p_{i,j})}(\{(x_1, \ldots, x_N)\}) = c\delta_{i'}(x_1)\delta_{i''}(x_N) \prod_{t=1}^{N-1} p_{x_t x_{t+1}} \tag{7}$$

with some positive constant c. **Theorem 4.1.**

Theorem 4.1. *For a Markov bridge and for the evolution laws given by $p \in \mathcal{P}_{\mathcal{Z}}$, cf. (1), (2), (7), the random transition count F is complete.*

Proof. Cf. Theorem 3.1. in [7]. The basic idea of this proof is to apply Lemma 3.6 and graph theory. □

We first present a reduced form of the main result which will be useful in its proving. Let $\mathcal{Z} \in S \times S$ be a fixed set. We always assume that $\mathcal{Z}$ satisfies (1), that is

$$\forall_{i \in S} \quad \exists_{j \in S} \quad (i, j) \notin \mathcal{Z}.$$

Recall that using $\mathcal{Z}$ one can naturally define the class S_0 of inessential states and classes of equivalence $S_1, \ldots, S_{\beta_0}$, $\beta_0 \in \mathbb{N}$, in essential states.

To prove the completeness of the statistic F, we need some more special properties of the set $\mathcal{Z}$. Let us remark that, by Example 2.3, such additional assumptions about the set $\mathcal{Z}$ are necessary.

Now we construct the set $\mathcal{Z}$ such that *the state space S will be the whole class of essential states* and we assume that there exists a permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of the set S, satisfying

$$\big\{(\pi_1, \pi_2), \ldots, (\pi_{n-1}, \pi_n), (\pi_n, \pi_1)\big\} \cap \mathcal{Z} = \emptyset. \tag{8}$$

Theorem 4.2. *Let $\mathcal{Z}$ satisfy (1) and (8). Let $(\mathbf{X}_{x_1,N}, \{P_{(p_{i,j})} :\ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$, with $\mathcal{P}_{\mathcal{Z}}$ given by (2), be the statistical space of all trajectories of Markov chains with the state space $S = \{1, \ldots, n\}$, a fixed initial state $x_1 = i'$, and the trajectory size $N \geq 2$. Then the random transition count F is complete.*

Proof. To show that the statistic F is complete, it is enough to prove that condition

$$\sum_f d(f) \cdot \xi(x_1, f) \cdot \prod_{i=1}^{n} \prod_{j=1}^{n} p_{i,j}^{f_{i,j}} = 0 \qquad \text{for each} \quad (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}} \tag{9}$$

implies that

$$d(f) = 0 \quad \text{for each } f,$$

where $\xi(x_1, f) = \#\{\mathbf{x} :\ F(\mathbf{x}) = f\}$ denote the number of corresponding trajectories. Applying assumptions of theorem, notations from Section 3, the equality (9) can be written as

$$\sum_{f'} d(f') \cdot W_{f'}(p) = 0,$$

for any $p = (p_{i,j})_{1 \leq i \leq n,\ 1 \leq j \leq m(i),\ j \neq i+1 \ (\text{mod } n)}$,
where

$$W_{f'}(p) = \prod_{i=1}^{n} \Bigg(\prod_{\substack{j=1 \\ j \neq i+1}}^{m(i)} p_{i,j}^{f'_{i,j}} \cdot \Big(1 - \sum_{\substack{j=1 \\ j \neq i+1}}^{m(i)} p_{i,j}\Big)^{\phi_i(f')} \Bigg).$$

Thus Lemma 3.6 completes the proof. □

In order to obtain completeness of F in a more general case it is necessary to make some extra assumptions about the set $\mathcal{Z}$ and S . Let $\mathcal{Z} \subset S \times S$ be a fixed set satisfying (1), that is

$$\forall_{i \in S} \ \exists_{j \in S} \quad (i, j) \notin \mathcal{Z}.$$

Let S_0 denote the class of inessential states and let $S_1, \ldots, S_{\beta_0},\ \beta_0 \in \mathbb{N}$ denote classes of equivalence in essential states.
Now we formulate assumptions about S:
(I) for each $\beta,\ 1 \leq \beta \leq \beta_0$ there exists a permutation $(i_1^\beta, \ldots, i_{n(\beta)}^\beta)$ of set S_β such that

$$\big\{(i_1^\beta, i_2^\beta), \ldots, (i_{n(\beta)-1}^\beta, i_{n(\beta)}^\beta), (i_{n(\beta)}^\beta, i_1^\beta)\big\} \cap \mathcal{Z} = \emptyset;$$

(II) for each $\beta,\ 1 \leq \beta \leq \beta_0$ there exists exactly one pair $(i_\beta, j_\beta) \in S_0 \times S_\beta$ such that

$$(i_\beta, j_\beta) \notin \mathcal{Z}.$$

Fix $\beta,\ 1 \leq \beta \leq \beta_0$. Our statistical space $(\mathbf{X}_{\beta,N},\ \{P^{\beta,N}{}_{(p_{i,j})} :\ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$ is defined as follows:
(i) the space $\mathbf{X}_{\beta,N}$ of trajectories $(x_1, \ldots, x_N)$ is determined by a fixed length N, a fixed initial state $x_1 = i'$ and a fixed essential state class S_β such that the final state x_N belongs to S_β;

(ii) the probability distribution

$$P^{\beta,N}{}_{(p_{i,j})}(\{\mathbf{x}\}) = p_{x_1,x_2} \cdot \ \dots \ \cdot p_{x_{N-1},x_N} \qquad \text{for} \qquad \mathbf{x} = (x_1, \dots, x_N) \in \mathbf{X}_{\beta,N}.$$

Without loss of generality we adopt $\beta = 1$.

Before we present our main result, let us observe that the random transition count F is a complete statistic if aditionally

(III) there exists a permutation $(i_1^0, \dots, i_{n(0)}^0)$ of the class S_0 of inessential states such that

$$\left\{(i_1^0, i_2^0), \dots, (i_{n(0)-1}^0, i_{n(0)}^0), (i_{n(0)}^0, i_1^0)\right\} \cap \mathcal{Z} = \emptyset.$$

Theorem 4.3. *Let $\mathcal{Z}$ satisfy (1) and conditions (I)-(III), and let the statistical space $(\mathbf{X}_{1,N}, \{P^{1,N}_{(p_{i,j})} : \ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$ as above be given. Then the random transition count F is complete.*

Proof. We only give the main ideas of the proof. Assume first that $x_1 = i' \in S_1$ is an essential state. Then one can assume that the space S is one class of essential states. Theorem 4.2 completes the proof in this case.

Now, let $x_1 = i' \in S_0$ be an inessential state. Let us consider a trajectory $\mathbf{x} \in \mathbf{X}_{1,N}$. Denote $t(\mathbf{x}) < N$ the number of steps on inessential states, so we perform $N-1-t(\mathbf{x})$ steps on S_1.

According to (9) we have

$$\text{(10)} \qquad \forall_{p \in \mathcal{P}_{\mathcal{Z}}} \left(\sum_{f=(f_{i,j})} \left(d(f) \cdot \xi(x_1, f) \cdot \prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'',i'''} \cdot \prod_{i,j \in S_1} p_{i,j}^{f_{i,j}} \right) = 0 \right).$$

By condition (III), Lemma 3.4 and Corollary 3.5, the factor

$$\prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'',i'''}$$

can be written as

$$\text{(11)} \qquad \begin{aligned} &\prod_{\substack{i \in S_0 \\ i \neq i''}} \ \prod_{\substack{j \in S_0 \\ j \neq i+1 (\bmod n(0))}} p_{i,j}^{\bar{f}_{i,j}} \cdot \Bigg(1 - \sum_{\substack{j \in S_0 \\ j \neq i+1 (\bmod n(0))}} p_{i,j}\Bigg)^{\psi_i(\bar{f})} \cdot \\ &\qquad \cdot \prod_{j \in S_0} p_{i'',j}^{\bar{f}_{i'',j}} \cdot \Bigg(1 - \sum_{j \in S_0} p_{i'',j}\Bigg)^{\psi_{i''}(\bar{f})}, \end{aligned}$$

obviously, $p_{i'',i'''} = 1 - \sum_{j \in S_0} p_{i'',j}$ and we have put $\psi_{i''}(\bar{f}) = 1$.

Observe that by a suitable change of notation the product (11) can be written as some polynomials $W_m(u)$ described in Section 3. Thus Lemma 3.6 and Theorem 4.2 completes the proof, because in the class S_1, the number of steps on essential states is known and a fixed essential state $i''' = j_1$ (cf. (II)) is the initial state. $\square$

We can now formulate our main result.

Theorem 4.4. *Let $\mathcal{Z} \subset S \times S$ satisfy (1) and conditions (I), (II). Let*

$$(\mathbf{X}_{1,N},\ \{P^{1,N}_{(p_{i,j})}:\ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$$

be the statistical space. Then the random transition count F is complete.

Proof. For $x_1 \in S_1$ Theorem 4.2 completes proof.
Now, assume that $x_1 = i' \in S_0$. We want to show that the statistic F is complete, that is that the condition (9) implies that $d(f) = 0$ for each f.

Let us consider a trajectory $\mathbf{x} \in \mathbf{X}_{1,N}$. Let $t(\mathbf{x})$ denote the number of steps on inessential states in the class S_0. For the first part $(x_1, \dots, x_{t(\mathbf{x})})$ of trajectory $\mathbf{x}$ we will consider an oriented graph $(\bar{S}_0,\ U)$, $U \subset \bar{S}_0 \times \bar{S}_0$ defined by $\mathcal{Z}$, i', i'', that is $\bar{S}_0 = \bigcup\{x_1, \dots, x_{t(\mathbf{x})}\}$; $U = \bigcup\{(x_1, x_2), \dots, (x_{t(\mathbf{x})-1}, x_{t(\mathbf{x})})\}$, with the unions taken for all $\mathbf{x}$ satisfying $x_1 = i'$, $x_{t(\mathbf{x})} = i''$ and $(x_t, x_{t+1}) \notin \mathcal{Z}$, $t = 1, \dots, t(\mathbf{x}) - 1$. By Lemma 3.8 there exists a tree $(\bar{S}_0, W)$, $W \subset U$ with a root i''. Next applying Lemma 3.9 and putting $p_{i'',i'''} = 1 - \sum_{j \in \bar{S}_0} p_{i'',j}$, $f_{i'',i'''} = \Phi_{i''}(m\,|_{U\setminus W}) = 1$, where $m = f\,|_U$,we can write the factor

$$\prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'',i'''}$$

in (10) in the form

$$\begin{aligned}
&\prod_{\substack{i \in \bar{S}_0 \\ i \neq i''}} \prod_{\substack{j \in \bar{S}_0 \\ (i,j) \in U\setminus W}} p_{i,j}^{m_{i,j}} \cdot \Big(1 - \sum_{\substack{j \in \bar{S}_0 \\ (i,j) \in U\setminus W}} p_{i,j}\Big)^{\Phi_i(m|_{U\setminus W})} \cdot \\
&\cdot \prod_{j \in \bar{S}_0} p_{i'',j}^{m_{i'',j}} \cdot \Big(1 - \sum_{\substack{j \in \bar{S}_0 \\ (i,j) \in U\setminus W}} p_{i'',j}\Big)^{\Phi_{i''}(m|_{U\setminus W})},
\end{aligned} \tag{12}$$

for a system of functions $\Phi_i,\ i \neq i''$, on the space $\{m\,|_{U\setminus W}\}$.
The set $U \setminus W$ can be written as

$$U \setminus W = \{(i,j):\quad i \in \bar{S}_0,\ j \in \bar{S}_0^i\},$$

where

$$\bar{S}_0^i = \{j \neq i + 1(\mod \bar{n}(0))\} \text{ for } i \in S_0 \text{ and } i \neq i'',$$
$$\bar{S}_0^{i''} = \bar{S}_0.$$

The rest of the proof is analogical to that of Theorem 4.3. □

Because the sufficency of F is obvious, so from Bahadur's Theorem [8] we have a

Conclusion. F is the minimal sufficient statistic.

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ZUPEŁNE STATYSTYKI DOSTATECZNE DLA ŁAŃCUCHÓW MARKOWA

S t r e s z c z e n i e

Praca ta prezentuje zupełność statystyk dostatecznych dla łańcuchów Markowa o skończonej przestrzeni stanów S dla pewnej naturalnej podprzestrzeni przestrzeni wszystkich macierzy prawdopodobieństw przejść. Statystyka będąca macierzą ilości przejść może być zupełną statystyką dostateczną, ale przy pewnych dodatkowch założeniach. Odpowiednie przykłady pokazują, że założenia te są konieczne. Praca jest próbą podsumowania najważniejszych wyników.

Słowa kluczowe: łańcuch Markowa, macierz ilości przejść, statystyka zupełna, statystyka dostateczna