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Waldemar Cieślak, Witold Mozgawa, and Paweł Wlaź

ON THE CLOSEST DISTANCE BETWEEN A POINT AND A CONVEX BODY

Summary

On a strictly convex curve we find explicitly a point realizing the shortest distance to a given point lying in the exterior of our curve. The result uses a support function of a strictly convex plane curve and can have practical applications. Moreover, we determine one of tangent lines to a strictly convex curve C passing through a given point lying in the exterior of C.

Keywords and phrases: strictly convex curve, support function, computational geometry

1. Introductory facts

In this paper we fix a strictly convex body in the plane and a point in its exterior. We investigate the following problem: find the point on the boundary of the fixed body realizing the minimal distance to the given point. Properties of proposed method are stated as a theorem, but we also concentrate on practical aspects of computations, giving algorithm that can be applied for exact or approximate solution of the problem.

Let C be a plane closed strictly convex curve which the origin lies in the region bounded by C. We denote by p the support function of C with respect to the origin. The support function p is differentiable and the parametrization of C in terms of this function is given by

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} = p(t)\cos t - \dot{p}(t)\sin t + i\,[p(t)\sin t + \dot{p}(t)\cos t]\,. \qquad (1)$$

The fundamental properties of support functions are presented in [2] and [5].

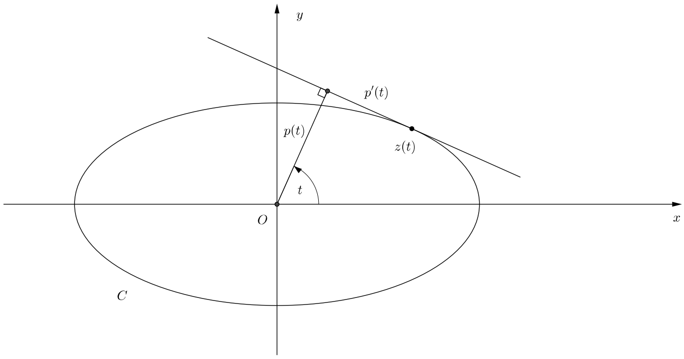

Fig. 1. Illustration of the support function $p(t)$ of the curve C with respect to the origin O, its derivative $\dot{p}(t)$ and the point $z(t)$.

We assume that $z(0)$ lies in the first quadrant. We find equation of support line to C passing through a given point $(b, 0)$, where $b > p(0)$. We introduce the notations as on Figure 2.

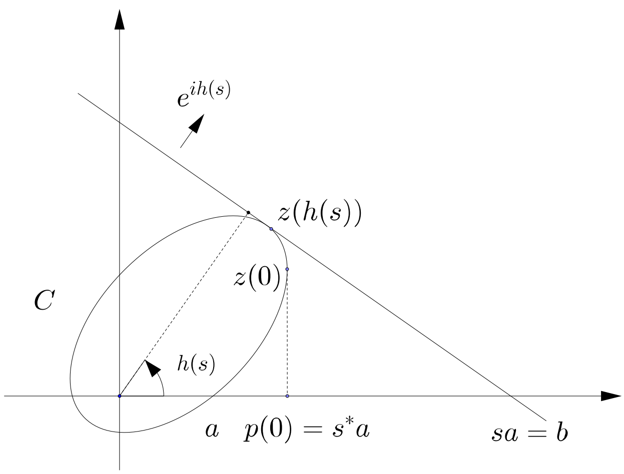

Fig. 2. Quantities a, $h(s)$, s^*a, sa.

We will consider h as a function of the variable $s \in (s^*, +\infty)$ with values in the interval $\left(0, \frac{\pi}{2}\right)$. For a fixed s we have

$$z(h(s)) + rie^{ih(s)} = sa$$

for some r. Hence we get

$$p(h(s)) = as \cos h(s). \tag{2}$$

Let

$$f(u) = \frac{p(u)}{a \cos u} \text{ for } u \in \left(0, \frac{\pi}{2}\right). \tag{3}$$

It is easy to see that

$$\begin{cases} \dot{f}(u) = \frac{\operatorname{Im} z(u)}{a \cos^2 u}, \\ f(0) = \frac{p(0)}{a}, \qquad \dot{f}(0) = \frac{\dot{p}(0)}{a}. \end{cases} \tag{4}$$

Our assumptions imply that $z(u)$ for $u \in \left(0, \frac{\pi}{2}\right)$ lies in the upper half-plane. Thus, we have $\operatorname{Im} z(u) > 0$ for $u \in \left(0, \frac{\pi}{2}\right)$, and f is a strictly increasing function.

We note that the condition (2) can be rewritten in the form $f \circ h = \mathrm{id}$. The function f is invertible, so we have

$$h = f^{-1}. \tag{5}$$

If $b = sa$ then our support line has the following equation

$$x + y \tan f^{-1}\left(\frac{b}{a}\right) - b = 0. \tag{6}$$

Example 1.1. Let $r > 8$ and $p(t) = r - \cos 3t$. We have $p(t) > 0$, $p(t + 2\pi) = p(t)$ and $p(t) + \ddot{p}(t) = r + 8\cos 3t > 0$. Thus p is a support function of some strictly convex curve C. We note that $z(0) = r - 1$, so $a = r - 1$ and $z(u)$ for $u \in (0, \frac{\pi}{2})$ lies in the upper half-plane. The function f in our case has the form

$$f(u) = \frac{r - \cos 3u}{(r-1)\cos u}.$$

We find the inverse function f^{-1} to f. If we set

$$s = \frac{r - \cos 3u}{(r-1)\cos u}$$

and

$$v = \cos u$$

then we have the following equation of the third degree

$$4v^3 + (-3 + (r-1)s)v - r = 0.$$

This equation has exactly one solution $v(s)$ in the interval $(0, 1)$, namely

$$v(s) = \frac{-3^{2/3} rs + 3^{2/3}(s+3) + \sqrt[3]{3}\left(9r + \sqrt{3}\sqrt{((r-1)s-3)^3 + 27r^2}\right)^{2/3}}{6\sqrt[3]{9r + \sqrt{3}\sqrt{((r-1)s-3)^3 + 27r^2}}}.$$

Thus the inverse function to f is given by

$$\tan^2 f^{-1}(s) = \frac{1}{v(s)^2} - 1$$

so our support line of C passing through a point $(b,0)$ has the following equation

$$x+y\sqrt{\frac{1}{v\left(\frac{b}{r-1}\right)^2}-1}-b=0.$$

2. Main result

In this section we begin by an auxiliary lemma.

Lemma 2.1. *Let C be a strictly convex curve given by* (1) *and $a=z(t^*)>0$. If C satisfies the condition*

$$\operatorname{Im} z(0)<0, \tag{7}$$

then the function $Q\colon (0,t^*)\to\mathbb{R}$ given by the formula

$$Q(u)=-\frac{\dot p(u)}{a\sin u} \tag{8}$$

is positive-valued and strictly decreasing. If C satisfies the condition

$$\operatorname{Im} z(0)>0, \tag{9}$$

then the function $Q^0\colon\ (t^*,2\pi)\to\mathbb{R}$ given by the formula

$$Q^0(v)=-\frac{\dot p(v)}{a\sin v} \tag{10}$$

is positive-valued and strictly increasing.

Proof. We may assume that $a=1$ and we prove the first part of the lemma since the second one can be proved similarly. Note that

$$Q(u)\cos u=p(u)-\frac{\operatorname{Im} z(u)}{\sin u}. \tag{11}$$

If $u\in(0,t^*)$, then $\operatorname{Im} z(u)<0$ and Q is a positive-valued function. Now, we prove that Q is strictly decreasing. Let $r,s\in(0,t^*)$ and $r<s$. Obviously, we have

$$\begin{cases} p(r)\cos r-\dot p(r)\sin r>p(s)\cos s-\dot p(s)\sin s\\ -p(r)\sin r-\dot p(r)\cos r>-p(s)\sin s-\dot p(s)\cos s. \end{cases} \tag{12}$$

It follows immediately from (12) that

$$p(r)\sin(s-r)-\dot p(r)\cos(s-r)>-\dot p(s) \tag{13}$$

for arbitrary $r,s\in(0,t^*)$ such that $r<s$. Now, let's assume that there exist $\hat s,\hat r\in(0,t^*)$ such that $\hat r<\hat s$ and $Q(\hat r)\le Q(\hat s)$, i.e.

$$\dot p(\hat r)\sin\hat r\cos\hat u+\dot p(\hat r)\cos\hat r\sin\hat u\ge\dot p(\hat s)\sin\hat r, \tag{14}$$

where $\hat u=\hat s-\hat r>0$. On the other hand from (13) we have

$$p(\hat r)\sin\hat u-\dot p(\hat r)\cos\hat u>-\dot p(\hat s). \tag{15}$$

Multiplying both sides of the inequality (15) by $\sin\hat{r}$ and then adding to (14) we get

$$[p(\hat{r})\sin\hat{r} + \dot{p}(\hat{r})\cos\hat{r}]\sin\hat{u} > 0,$$

what means that $\operatorname{Im} z(\hat{r}) > 0$ and we get a contradiction.

□

Now, we use this lemma to show our main result.

Theorem 2.2. *Let C be a strictly convex curve given by (1) and $a = z(t^*) > 0$. If $b > a$ and $\operatorname{Im} z(0) < 0$ then the point $z\left(Q^{-1}\left(\frac{b}{a}\right)\right)$, where Q is given by (8), realizes the shortest distance between $(b, 0)$ and C.*

Proof.

Let C satisfy the condition (7) and $b > a$. We find a point on C realizing the shortest distance between the point $(b, 0)$ and C. We use notations as on Figure 3.

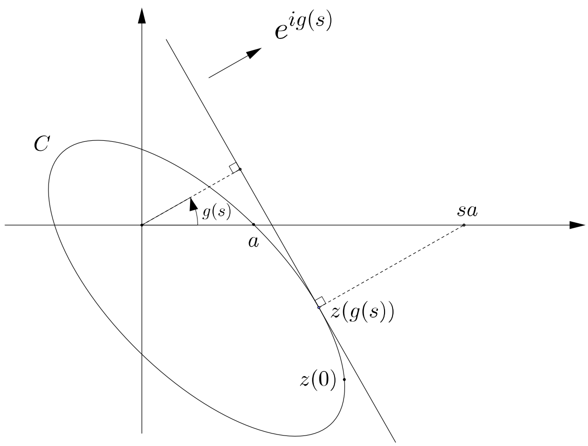

Fig. 3. Quantities a, $g(s)$, $z(g(s))$, sa.

We will treat g as a function of the variable $s \in (1, +\infty)$ with values in $(0, t^*)$. It is clear that the vectors $e^{ig(s)}$ and $z(g(s)) - sa$ have to be collinear, i.e.

$$z(g(s)) - sa = re^{ig(s)}$$

for some $r \in \mathbb{R}$. Hence we get

$$-\dot{p}(g(s)) = sa\sin g(s). \tag{16}$$

The formula (16) can be rewritten in the following form

$$Q \circ g = \operatorname{id}.$$

In view of Lemma 2.1 we have $g = Q^{-1}$. Thus we proved our main theorem. □

Example 2.3. In this example we illustrate our above method and show its simplicity. We will determine the distance from the point $P\left(3\sqrt{3},\frac{9}{2}\right)$ to the ellipse $\frac{x^2}{4}+y^2=1$. Such ellipse has the support function in Oxy coordinates given by the formula

$$p(t)=\sqrt{4\cos^2 t+\sin^2 t}$$

and in fact the situation is illustrated on Figure 1. Then the formula (1) becomes

$$z(t)=\frac{12+4\cos 2t+\sin 2t}{4\sqrt{4\cos t+\sin t}}+\frac{3-\cos 2t+4\sin 2t}{4\sqrt{4\cos t+\sin t}}i.$$

Note that in the notations of Figure 4 we have that the point C has in the system Oxy the coordinates $\left(1,\frac{\sqrt{3}}{2}\right)$. We rotate the coordinate system counter clockwise about the angle ω between the radius vector of the point P and the axis Ox. Using the coordinates of P we get immediately that

$$\tan\omega=\frac{\sqrt{3}}{2},$$

thus the equation of our ellipse in new coordinates $Ox'y'$ becomes

$$16\left(x'\right)^2+12\sqrt{3}x'y'+19\left(y'\right)^2=28.$$

Evidently, the points C and P lie now on the Ox' axis and have the new coordinates $\left(\frac{\sqrt{7}}{2},0\right)$ and $\left(\frac{3\sqrt{21}}{2},0\right)$, respectively. Hence $a=|OC|=\frac{\sqrt{7}}{2}$ and $b=|OP|=\frac{3\sqrt{21}}{2}$. The support function $q(t)$ of our ellipse in new coordinate system has the form

$$q(t)=p(t+\omega)=\sqrt{4\cos^2(t+\omega)+\sin^2(t+\omega)}.$$

Next from Figure 4 we read that

$$\tan\left(t^*+\omega\right)=2\sqrt{3},$$

hence

$$\tan t^*=\frac{8\sqrt{3}}{3}\approx 4.6188\ldots$$

and $t^*\approx 1.35758$. Thus from Lemma 2.1 we get that the function $Q(u)=-\frac{\dot{q}(u)}{a\sin u}$ is positive-valued and strictly decreasing on interval $(0,t^*)$. So we need to determine $u=Q^{-1}\left(\frac{b}{a}\right)$. We solve the equation $Q(u)=\frac{b}{a}$ i.e.

$$-\frac{\dot{q}(u)}{a\sin u}=\frac{b}{a}$$

or

$$\dot{p}(u+\omega)=-b\sin u.$$

Inserting all necessary data, $\cos\omega=\frac{2}{\sqrt{7}}$ and $\sin\omega=\frac{\sqrt{3}}{\sqrt{7}}$ and substituting $x=\tan u$ we arrive to the quartic equation

$$2304x^4-1748\sqrt{3}x^3+2885x^2-16\sqrt{3}x-48=0.$$

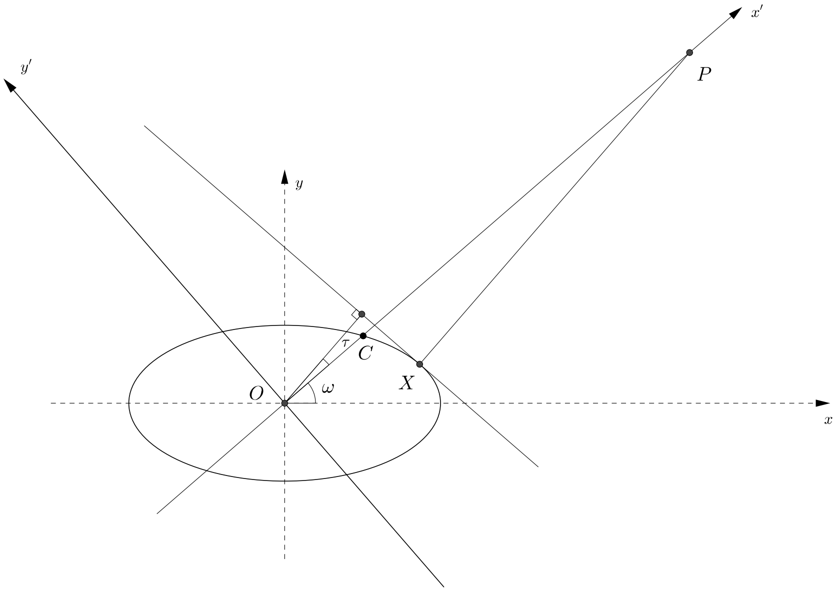

Fig. 4. Realization of the shortest distance from P to our ellipse.

Let $\Omega(x) = 2304x^4 - 1748\sqrt{3}x^3 + 2885x^2 - 16\sqrt{3}x - 48$. It is easy to see that $\Omega(0) < 0$ and $\Omega(1) > 0$, moreover $1 = \tan\frac{\pi}{4} < \tan t^*$ and Q is strictly decreasing thus there exists exactly one solution $x_0 \in (0, t^*)$ which gives the closest point on the ellipse to P. One can check directly that $x_0 = \frac{1}{4\sqrt{3}}$. Thus in Oxy coordinates we have that $X = z(\omega + \arctan x_0) = \sqrt{3} + \frac{1}{2}i$. The point $\left(\sqrt{3}, \frac{1}{2}\right)$ is the searched point realizing the claimed minimum, as we see on Figure 4.

3. Computer approximation

In general it is not easy to obtain the inverse of the function Q in Theorem 2.2, but we can approximate its inverse, what gives us the possibility of finding approximation that realizes the shortest distance between a given point and a strictly convex curve.

An algorithm deploying ideas previously introduced in this paper could be divided into two parts, the first one (part A) is to be done once for a given convex set, the second (part B) – once for a given point.

A-1. Let $p(t)$ be the support function of our convex set bounded by C.

If $p(t)$ cannot be obtained as an analytical expression for some reasons it can be approximated in various ways. For example one can set large integer M, set

$h = 2\pi/M$ and for all $t \in \{0, h, 2h, \ldots, (M-1)h\}$ calculate

$$p(t) = \max_{(x,y)\in C} (x \cos t + y \sin t). \tag{17}$$

How easily the value in (17) can be obtained depends on the nature of C, in most practical cases, like C being a set of Bézier curves, it is straightforward. For example if a Bézier curve is cubic, then the first derivative (with respect to s) of

$$x(s) \cos t + y(s) \sin t, \quad s \in (0, 1)$$

is a polynomial in s of the second order.

But in general finding the support function (or its approximation) is beyond the scope of this paper, and we just assume it is calculated. Once for a given C.

A-2. Let $\dot{p}(t)$ be a derivative of $p(t)$. If we do not have an analytical expression for $p(t)$, we can, for $t \in \{0, h, 2h, \ldots, (M-1)h\}$ approximate $\dot{p}(t)$ as

$$(p(t+h) - p(t-h))/(2h).$$

Again, obtaining $\dot{p}(t)$ is done once for given C.

A-3. Having p and $\dot{p}$ we may find all t^* for all $t \in \{0, h, 2h, \ldots, (M-1)h\}$, such that

$$\arg(z(t^*)) \approx t, \qquad t, t^* \in \{0, h, 2h, \ldots, (M-1)h\}.$$

Having these functions (or their approximations) prepared, we may proceed with main algorithm.

Suppose we have a point $be^{i\alpha}$, outside of our C, we want to find the closest point on C.

B-1. Obtain t^* for $t \approx \alpha$ (prepared in *A*-3).

B-2. If $\dot{p}(\alpha) = 0$ (or, in other words, $t^* = \alpha$) then our optimal point is $p(\alpha)e^{i\alpha}$, the algorithm stops.

B-3. If $\dot{p}(\alpha) < 0$ then let $L = \alpha$, let $R = t^*$ (and if R happened to be smaller than L then advance R by 2π.

B-4. If $\dot{p}(\alpha) > 0$ then let $L = t^*$, let $R = \alpha$ (and if L happened to be greater than R then reduce L by 2π.

B-5. Now solve equation

$$\frac{-\dot{p}(u)}{\sin(u-\alpha)} = b, \qquad u \in (L, R). \tag{18}$$

Lemma 2.1 guarantees uniqueness of the solution, and that the left side of (18) is strictly monotonic, so it can be solved by simple numerical methods, for example by bisection.

B-6. Due to theorem 2.2 the optimal point is

$$z(u) = p(u)e^{iu} + \dot{p}(u)ie^{it}.$$

Example 3.1. Let us consider a set described by two cubic Bézier curves illustrated on Figure 5. The first curve is defined by points $(-5,0)$, $(-5,8)$, $(4,2)$, $(6,0)$. The second one is defined by $(-5,0)$, $(-4,-3)$, $(5-1)$, $(6,0)$. Curves are illustrated with solid lines. Then we have 36 points of the form $7 \cdot e^{i2\pi/36}$ and for each we draw its closest point on C, connected with dashed segment. All calculations for drawing the picture were done exactly with the described algorithm.

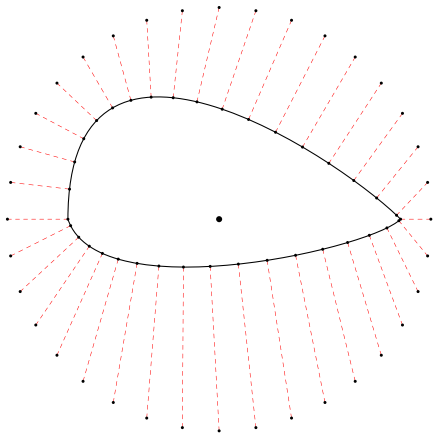

Fig. 5. Finding the closest points on Bézier curves.

Example 3.2. Another example is a convex set bounded by a curve

$$|x|^3 + |y|^3 = 1.$$

Again, applying the above algorithms gives us the closest points to particular 36 points of the form $2 \cdot e^{i2\pi/36}$, illustrated on Figure 6.

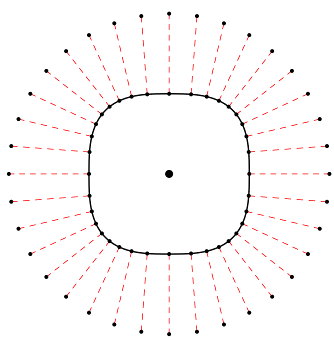

Fig. 6. Finding the closest points on $|x|^3 + |y|^3 = 1$.

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Lublin University of Technology
Department of Applied Mathematics
Nadbystrzycka 40, PL-20-618 Lublin
Poland
E-mail: izacieslak@wp.pl
p.wlaz@pollub.pl

Maria Curie-Skłodowska University
Institute of Mathematics
pl. M. Curie-Skłodowskiej 1, PL-20-031 Lublin
Poland
E-mail: mozgawa@poczta.umcs.lublin.pl

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O NAJMNIEJSZEJ ODLEGŁOŚCI MIĘDZY PUNKTEM A CIAŁEM WYPUKŁYM

S t r e s z c z e n i e

Na ściśle wypukłej krzywej znajdujemy punkt realizujący najmniejszą odległość do danego punktu leżącego w zewnętrzu krzywej. Otrzymane rezultaty wykorzystują funkcję podparcia ściśle wypukłej krzywej i mogą mieć praktyczne zastosowania. Ponadto, wyznaczamy jedną ze stycznych do ściśle wypukłej krzywej C przechodzącą przez dany punkt leżący w zewnętrzu C.

Słowa kluczowe: krzywa ściśle wypukła, funkcja podparcia, geometria obliczeniowa