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Armen Grigoryan and Dariusz Partyka

LUSIN'S TYPE THEOREMS FOR LIMITABLE FUNCTIONS

Summary

In this paper we introduce limitable functions (the ones which have a limit at each accumulation point of their domains) and prove Lusin's type theorems for such functions. We also give a characterization of limitable functions.

Keywords and phrases: limiting valued function, Lindelöf property, Lusin's theorem

1. Introduction

Let (X, ρ) be a metric space and let A be a subset of X. We denote by $\mathrm{cl}(A)$ and $\mathrm{fr}(A)$ the closure and boundary of A, respectively. The set of all accumulation points of A will be denoted by A^d. If $x_0 \in X$ then the open ball at the center x_0 and radius $r > 0$ we denote by $B(x_0, r)$, i.e., $B(x_0, r) = \{x \in X : \rho(x_0, x) < r\}$.

Throughout the paper we assume that (X_1, ρ_1) and (X_2, ρ_2) are metric spaces and Ω is a non-empty subset of X_1. Moreover, unless otherwise stated the sets $\mathrm{cl}(A)$, $\mathrm{fr}(A)$ and A^d of a set $A \subset X_1$ are understood in the space (X_1, ρ_1).

Definition 1.1. A function $f \colon \Omega \to X_2$ is said to be *limitable* if $\Omega^d = \emptyset$ or $\Omega^d \neq \emptyset$ and for each $x \in \Omega^d$ there exists the limit $\lim_{t\to x} f(t)$.

The class of all limitable functions $f \colon \Omega \to X_2$ will be denoted by $\mathcal{L}(\Omega)$. It is easy to see that if the metric space (X_2, ρ_2) is complete and $f \colon \Omega \to X_2$ is uniformly continuous, then $f \in \mathcal{L}(\Omega)$. In particular, $f \in \mathcal{L}(\Omega)$ provided f is Hölder continuous in Ω, i.e., there exist $C \geq 0$ and $\alpha \in (0; 1]$ such that

$$\rho_2(f(t_1), f(t_2)) \leq C\rho_1(t_1, t_2)^\alpha \quad \text{for } t_1, t_2 \in \Omega. \tag{1.1}$$

With each $f \in \mathcal{L}(\Omega)$ we assign the function $\hat{f}\colon \operatorname{cl}(\Omega) \to X_2$ defined by the formula

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega \setminus \Omega^d, \\ \lim_{t\to x} f(t) & \text{if } x \in \Omega^d. \end{cases} \tag{1.2}$$

Note that $\hat{f}$ coincides with f on the set of isolated points of Ω, provided $\Omega \setminus \Omega^d \neq \emptyset$.

Remark 1.2. (cf. Problem 8 in [12], Ch. III, §2). If $f \in \mathcal{L}(\Omega)$, then the function $\hat{f}\colon \operatorname{cl}(\Omega) \to X_2$ defined by (1.2) is continuous. In order to prove this fact fix $x \in \operatorname{cl}(\Omega)$. If $x \in \Omega \setminus \Omega^d$, then the continuity of $\hat{f}$ at x is obvious. Suppose now that $x \in \Omega^d$. Then $\hat{f}(x) = \lim_{t\to x} f(t)$. Fix $\varepsilon > 0$. Then there exists $\delta_\varepsilon > 0$ such that

$$\rho_2\left(f(t), \hat{f}(x)\right) < \varepsilon/2 \quad \text{for } t \in \Omega,\ 0 < \rho_1(t,x) < 2\delta_\varepsilon. \tag{1.3}$$

Now fix $t \in B(x,\delta_\varepsilon) \setminus \{x\}$. If $t \in \Omega \setminus \Omega^d$ then by (1.3) we have

$$\rho_2\left(\hat{f}(t), \hat{f}(x)\right) = \rho_2\left(f(t), \hat{f}(x)\right) < \varepsilon.$$

If $t \in \Omega^d$, then $\hat{f}(t) = \lim_{t'\to x} f(t')$, and so there exists $\delta'_\varepsilon \in (0;\delta_\varepsilon)$ and $t' \in \Omega \cap B(t,\delta'_\varepsilon) \setminus \{t\}$ such that

$$\rho_2\left(f(t'), \hat{f}(t)\right) < \varepsilon/2. \tag{1.4}$$

Since

$$\rho_1(t',x) \leq \rho_1(t',t) + \rho_1(t,x) < \delta'_\varepsilon + \delta_\varepsilon < 2\delta_\varepsilon,$$

we deduce from (1.3) and (1.4) that

$$\rho_2\left(\hat{f}(t), \hat{f}(x)\right) \leq \rho_2\left(\hat{f}(t), f(t')\right) + \rho_2\left(f(t'), \hat{f}(x)\right) < \varepsilon.$$

Consequently,

$$\rho_2\left(\hat{f}(t), \hat{f}(x)\right) < \varepsilon \quad \text{for } t \in B(x,\delta_\varepsilon) \cap \operatorname{cl}(\Omega),$$

which is the desired conclusion.

Limitable functions appear naturally in various aspects of complex analysis. In what follows we present few examples dealing with mappings in the complex plane $\mathrm{E}(\mathbb{C}) := (\mathbb{C}, \rho_e)$ where ρ_e is the standard Euclidean metric.

Example 1.3. Suppose that (X_1,ρ_1) and (X_2,ρ_2) coincide with $\mathrm{E}(\mathbb{C})$ and set $\Omega := B(0,1)$. Let $f\colon \Omega \to \mathbb{C}$ be a K-quasiconformal mapping for some $K \geq 1$. By the Riemann mapping theorem there exists a conformal mapping f_0 of Ω onto $f(\Omega)$ such that $f_0(0) = f(0)$; cf. [10, p. 283]. Then $f_1 := f_0^{-1} \circ f$ is a K-quasiconformal self-mapping of the unit disk Ω keeping the origin fixed. From Mori's theorem it follows that f_1 satisfies the Hölder condition (1.1) with $C := 16$ and $\alpha := 1/K$; cf. [1, p. 47] and [6, pp. 66–67]. Therefore $f_1 \in \mathcal{L}(\Omega)$. If additionally each point of $\operatorname{fr}(f(\Omega))$ is a simple boundary point of the domain $f(\Omega)$, then $f_0 \in \mathcal{L}(\Omega)$, and so $f \in \mathcal{L}(\Omega)$; cf. [10, p. 290].

Example 1.4. Let (X_1, ρ_1), (X_2, ρ_2) and Ω be like in Example 1.3. Suppose that f is a conformal mapping of Ω onto $f(\Omega)$ bounded by a Dini-smooth Jordan curve. As shown by Warschawski, $f' \in \mathcal{L}(\Omega)$; cf. [9, p. 298].

Example 1.5. Let (X_1, ρ_1), (X_2, ρ_2) and Ω be like in Example 1.3. Suppose that f is a harmonic injective mapping of Ω onto itself. Then Choquet theorem implies that $f \in \mathcal{L}(\Omega)$; cf. [2], see also [3, Sect. 3.3].

Example 1.6. Let (X_1, ρ_1), (X_2, ρ_2) and Ω be like in Example 1.3. Suppose that $\varphi\colon \mathrm{fr}(\Omega) \to \mathbb{C}$ is a continuous function. Then $f := \mathrm{P}[\varphi] \in \mathcal{L}(\Omega)$, where $\mathrm{P}[\varphi]$ is the Poisson integral of φ, i.e.,

$$\mathrm{P}[f](z) := \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t}) \frac{1 - |z|^2}{|\mathrm{e}^{\mathrm{i}t} - z|^2} \mathrm{d}t \quad \text{for } z \in \Omega;$$

cf. e.g. [10, p. 233–234] and [5, Chap. I, Sect. 3].

In the next section we prove a crucial lemma which has its own interest. In Section 3 our main results relevant to Lusin's theorem are stated and proved. In the last section we give sufficient and necessary conditions for a function to be limitable.

2. An auxiliary lemma

We start with an auxiliary lemma, which plays an important role later on.

Lemma 2.1. *If* $\mathrm{cl}(\Omega)$ *is compact in* (X_1, ρ_1) *and* $f \in \mathcal{L}(\Omega)$*, then for each* $\varepsilon > 0$ *there exist* $\delta_\varepsilon > 0$ *and a finite set* $A_\varepsilon \subset \Omega^d$ *such that* $\Omega \setminus A_\varepsilon \neq \emptyset$ *and*

$$\rho_2\left(f(t_1), f(t_2)\right) < \varepsilon \quad \textit{for } t_1, t_2 \in \Omega \setminus A_\varepsilon\,,\ \rho_1(t_1, t_2) < \delta_\varepsilon. \tag{2.1}$$

Proof. Given $f \in \mathcal{L}(\Omega)$ fix $\varepsilon > 0$. Suppose first that $\Omega^d = \emptyset$. Then each point of Ω is isolated, and so $\mathrm{cl}(\Omega) = \Omega$. Since $\mathrm{cl}(\Omega)$ is a compact set in (X_1, ρ_1), it follows that Ω is finite. Therefore the property (2.1) holds with $A_\varepsilon := \emptyset$ and $\delta_\varepsilon := \frac{1}{2}\min\left(\{\rho_1(t_1, t_2)\colon t_1, t_2 \in \Omega\,,\ t_1 \neq t_2\}\right) > 0$.

Suppose now that $\Omega^d \neq \emptyset$. By Definition 1.1, for each $x \in \Omega^d$ there exists $\eta_{\varepsilon,x} > 0$ such that

$$\rho_2(f(t), \hat{f}(x)) < \frac{\varepsilon}{2} \quad \text{for } t \in \Omega \cap B(x, \eta_{\varepsilon,x}) \setminus \{x\}. \tag{2.2}$$

Obviously,

$$\mathrm{cl}(\Omega) = \Omega^d \cup \left(\Omega \setminus \Omega^d\right).$$

Let $x \in \mathrm{cl}(\Omega)$. If $x \in \Omega \setminus \Omega^d$ then there exists $\eta_x > 0$ such that

$$B(x, \eta_x) \cap \Omega = \{x\}. \tag{2.3}$$

Otherwise $x \in \Omega^d$ and we take $\eta_x := \eta_{\varepsilon,x}$. The collection $\{B(x, \eta_x/2)\colon x \in \mathrm{cl}(\Omega)\}$ is an open cover of $\mathrm{cl}(\Omega)$. Hence, by the compactness of $\mathrm{cl}(\Omega)$, there exists a finite subset J_ε of $\mathrm{cl}(\Omega)$ such that

$$\mathrm{cl}(\Omega) \subset \bigcup_{x \in J_\varepsilon} B(x, \eta_x/2). \tag{2.4}$$

Define

$$A_\varepsilon := J_\varepsilon \cap \Omega^d \quad \text{and} \quad \delta_\varepsilon := \frac{1}{2} \min\left(\{\eta_x : x \in J_\varepsilon\}\right).$$

Since J_ε is a finite set, $\delta_\varepsilon > 0$.

Now fix arbitrary $t_1, t_2 \in \Omega \setminus A_\varepsilon$ such that $\rho_1(t_1, t_2) < \delta_\varepsilon$. By (2.4) there exists $x \in J_\varepsilon$ such that $t_1 \in B(x, \eta_x/2)$. By the definition of δ_ε,

$$\rho_1(t_2, x) \le \rho_1(t_2, t_1) + \rho_1(t_1, x) < \delta_\varepsilon + \eta_x/2 \le \eta_x,$$

which means that $t_1, t_2 \in B(x, \eta_x)$. If $x \in J_\varepsilon \setminus A_\varepsilon$ then by (2.3) we have $t_1 = t_2 = x$, and consequently

$$\rho_2(f(t_1), f(t_2)) = 0 < \varepsilon.$$

If $x \in J_\varepsilon \cap A_\varepsilon$ then by (2.2) we get

$$\rho_2(f(t_1), f(t_2)) \le \rho_2(f(t_1), \hat{f}(x)) + \rho_2(f(t_2), \hat{f}(x)) < \varepsilon,$$

because $t_1 \ne x$ and $t_2 \ne x$. This completes the proof. □

3. Lusin's type theorems

Recall that a set S is countable if $S = \emptyset$ or there exists an injective function $h\colon S \to \mathbb{N}$ (in particular a countable set can be finite). For every function $f\colon D \to Y$ and every set $\widetilde{D} \subset D$ the restriction of f to $\widetilde{D}$ will be denoted by $f|_{\widetilde{D}}$. One of the consequences of Lemma 2.1 is the following counterpart of Lusin's theorem for measurable functions, cf. [7], [8, p. 106], [10, p. 55].

Theorem 3.1. *If Ω is not countable, $\mathrm{cl}(\Omega)$ is compact in (X_1, ρ_1) and $f \in \mathcal{L}(\Omega)$, then there exists a countable subset A of the set Ω^d such that the following two conditions hold:*

(i) *$f|_{\Omega \setminus A}$ is uniformly continuous;*
(ii) *f is continuous at each point $x \in \Omega \setminus A$.*

Proof. Fix $f \in \mathcal{L}(\Omega)$ and consider the sequence $\mathbb{N} \ni n \mapsto \varepsilon_n := 1/n$. By Lemma 2.1 for each $n \in \mathbb{N}$ there exist $\tilde{\delta}_{\varepsilon_n} > 0$ and a finite set $A_{\varepsilon_n} \subset \Omega^d$ such that

$$\rho_2\left(f(t_1), f(t_2)\right) < \varepsilon_n \quad \text{for} \quad t_1, t_2 \in \Omega \setminus A_{\varepsilon_n},\ \rho_1(t_1, t_2) < \tilde{\delta}_{\varepsilon_n}. \tag{3.1}$$

Define

$$A := \bigcup_{n=1}^{\infty} A_{\varepsilon_n}.$$

The set A is countable as a sum of a countable family of finite sets. Now fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0} < \varepsilon$. Hence, by (3.1) we have

$$\rho_2\left(f(t_1), f(t_2)\right) < \varepsilon_{n_0} < \varepsilon \quad \text{for} \quad t_1, t_2 \in \Omega \setminus A_{\varepsilon_{n_0}},\ \rho_1(t_1, t_2) < \tilde{\delta}_{\varepsilon_{n_0}}.$$

Therefore

$$\rho_2\left(f(t_1), f(t_2)\right) < \varepsilon \quad \text{for} \quad t_1, t_2 \in \Omega \setminus A,\ \rho_1(t_1, t_2) < \tilde{\delta}_{\varepsilon_{n_0}},$$

which proves (i). In order to prove (ii) suppose that $x \in \Omega \setminus A$ is arbitrarily given. Define

$$\delta_x := \min\left(\{\tilde{\delta}_{\varepsilon_{n_0}}, \min\left(\{\rho_1(x, a) \colon a \in A_{\varepsilon_{n_0}}\}\right)\}\right).$$

For every $t \in \Omega$ such that $\rho_1(t, x) < \delta_x$ we see that

$$t \in \Omega \setminus A_{\varepsilon_{n_0}} \quad \text{and} \quad \rho_1(t, x) < \tilde{\delta}_{\varepsilon_{n_0}},$$

which gives, by the inclusion $\Omega \setminus A \subset \Omega \setminus A_{\varepsilon_{n_0}}$ and (3.1), that

$$\rho_2\left(f(t), f(x)\right) < \varepsilon_{n_0} < \varepsilon.$$

This completes the proof. □

Remark 3.2. Theorem 3.1 is trivially valid if Ω is countable. In such a case it is enough to put $A := \Omega \setminus \{x\}$ for an arbitrarily chosen point $x \in \Omega$.

Example 3.3. Consider the Riemann function $\varphi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) := \begin{cases} 1/n, & \text{if}\, x = m/n,\ n \in \mathbb{N},\ m \in \mathbb{Z},\ \gcd(n, m) = 1, \\ 0, & \text{if}\, x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Suppose that (X_1, ρ_1) and (X_2, ρ_2) coincide with $\mathrm{E}(\mathbb{R}) := (\mathbb{R}, \rho_e)$, where ρ_e is the usual Euclidean metric, and set $\Omega := [0; 1]$. Since $\mathbb{R}^d = \mathbb{R}$ and $\lim_{t \to x} \varphi(t) = 0$ for each $x \in \mathbb{R}$ we see that $f := \varphi|_\Omega \in \mathcal{L}(\Omega)$ and $\hat{f}(x) = 0$ for each $x \in \mathrm{cl}(\Omega) = \Omega$. The function f is "very discontinuous". Nevertheless, it is uniformly continuous on the set $\Omega \setminus \mathbb{Q}$. Moreover, this example shows that we cannot replace the word "countable" by "finite" in the conclusion of Theorem 3.1.

Recall that a set $A \subset X$ has a *Lindelöf* property in (X, ρ) if each open cover of A contains a countable subcover of A, cf. [11, p. 116], [4, p. 192]. From Theorem 3.1 we can drive a variant of Lusin's type theorem for limitable functions (see Theorem 3.4), where we replace the compactness of $\mathrm{cl}(\Omega)$ by the Lindelöf property of $\mathrm{cl}(\Omega)$ and add the locally compactness of (X_1, ρ_1).

Theorem 3.4. *If Ω is not countable, $\mathrm{cl}(\Omega)$ has a Lindelöf property in a locally compact metric space (X_1, ρ_1) and $f \in \mathcal{L}(\Omega)$, then there exists a countable subset A of the set Ω^d such that $f|_{\Omega \setminus A}$ is continuous.*

Proof. Fix $f \in \mathcal{L}(\Omega)$. Since the space (X_1, ρ_1) is locally compact, for each $x \in \mathrm{cl}(\Omega)$ there exists $r_x > 0$ such that $\mathrm{cl}(B(x, r_x))$ is compact in (X_1, ρ_1). The family $\mathcal{F} := \{B(x, r_x) \colon x \in \mathrm{cl}(\Omega)\}$ is an open cover of $\mathrm{cl}(\Omega)$. Since $\mathrm{cl}(\Omega)$ has a Lindelöf property, there exists a non-empty countable set N such that the subfamily $\{B(x_n, r_{x_n})\colon n \in N\}$ of $\mathcal{F}$ covers $\mathrm{cl}(\Omega)$. Set

$$r_n := r_{x_n} \text{ and } \tilde{\Omega}_n := B(x_n, r_n) \cap \Omega \quad \text{for } n \in N. \tag{3.2}$$

Obviously,

$$\bigcup_{n\in N} \tilde{\Omega}_n = \Omega.$$

Fix $n \in N$. By (3.2) we have

$$\mathrm{cl}(\tilde{\Omega}_n) = \mathrm{cl}(B(x_n, r_{x_n}) \cap \Omega) \subset \mathrm{cl}(B(x_n, r_n)) \cap \mathrm{cl}(\Omega) \subset \mathrm{cl}(B(x_n, r_n)).$$

Hence, by the compactness of $\mathrm{cl}(B(x_n, r_n))$, the closed set $\mathrm{cl}(\tilde{\Omega}_n)$ is compact in (X_1, ρ_1). Since $\tilde{\Omega}_n \subset \Omega$ we see that $\tilde{\Omega}_n^d \subset \Omega^d$ and $f|_{\tilde{\Omega}_n} \in \mathcal{L}(\tilde{\Omega}_n)$. Therefore, by Theorem 3.1 and Remark 3.2, there exists a countable set $A_n \subset \tilde{\Omega}_n^d \subset \Omega^d$ such that $\tilde{\Omega}_n \setminus A_n \neq \emptyset$ and the function $f|_{\tilde{\Omega}_n \setminus A_n}$ is uniformly continuous. Define

$$A := \bigcup_{n\in N} A_n.$$

The set A is countable as a union of countably many countable sets. Evidently $A \subset \Omega^d$. It remains to show that $f|_D$ is continuous where $D := \Omega \setminus A$. Let $x \in D$. Since $D \subset \bigcup_{n\in N} \left(\tilde{\Omega}_n \setminus A_n\right)$, there exists $n_0 \in N$ such that $x \in \tilde{\Omega}_{n_0} \setminus A_{n_0}$. By (3.2) we see that $x \in (B(x_{n_0}, r_{n_0}) \cap \Omega) \setminus A_{n_0}$. Setting $\delta_x := r_{n_0} - \rho_1(x, x_{n_0})$ we get $B(x, \delta_x) \subset B(x_{n_0}, r_{n_0})$, and so

$$B(x, \delta_x) \cap D \subset B(x_{n_0}, r_{n_0}) \cap (\Omega \setminus A_{n_0}) = \tilde{\Omega}_{n_0} \setminus A_{n_0}.$$

Hence, by the continuity of $f|_{\tilde{\Omega}_{n_0} \setminus A_{n_0}}$, the function $f|_{B(x,\delta_x)\cap D}$ is continuous. Now we have to prove that $f|_D$ is continuous at x. Fix $\varepsilon > 0$. By the continuity of $f|_{B(x,\delta_x)\cap D}$ at x there exists $\eta_{x,\varepsilon} > 0$ such that

$$\rho_2(f|_D(t), f|_D(x)) < \varepsilon \quad \text{for } t \in B(x, \delta_x) \cap D,\ \rho_1(t, x) < \eta_{x,\varepsilon}. \tag{3.3}$$

Set $\tilde{\delta}_{x,\varepsilon} := \min\left(\{\delta_x, \eta_{x,\varepsilon}\}\right)$. By (3.3), we have

$$\rho_2(f|_D(t), f|_D(x)) < \varepsilon \quad \text{for } t \in D,\ \rho_1(t, x) < \tilde{\delta}_{x,\varepsilon},$$

which means the continuity of $f|_D$ at x. Therefore the function $f|_D$ is continuous, and we are done. □

In the following variant of Lusin's type theorem we drop any condition on the set $\mathrm{cl}(\Omega)$. Instead we add the separability of the metric space (X_1, ρ_1).

Theorem 3.5. *If Ω is not countable, (X_1, ρ_1) is a separable and locally compact metric space and $f \in \mathcal{L}(\Omega)$, then there exists a countable subset A of the set Ω^d such that $f|_{\Omega\setminus A}$ is continuous.*

Proof. Let Ω and (X_1, ρ_1) satisfy the assertion of the theorem. We will prove that $\mathrm{cl}(\Omega)$ has a Lindelöf property in the space (X_1, ρ_1), as a closed subset of a separable space. This is a rather known fact (see, e.g., [4, p. 192]). However, to make the proof self-contained we give an explicit proof of this fact.

Let $\mathcal{R}$ be an open cover of $\mathrm{cl}(\Omega)$. Since $\mathrm{cl}(\Omega)$ is a closed set in (X_1, ρ_1), its complement to X_1, i.e., the set $X_1 \setminus \mathrm{cl}(\Omega)$ is open in (X_1, ρ_1). Denote by $\hat{\mathcal{R}} := \mathcal{R} \cup \{X_1 \setminus \mathrm{cl}(\Omega)\}$. Obviously, $\hat{\mathcal{R}}$ is an open cover of X_1. Since (X_1, ρ_1) is separable, there exists a non-empty countable set A such that $\mathrm{cl}(A) = X_1$. Define

$$A^* := \left\{(a, m) \in A \times \mathbb{N}\colon \text{there exists } U \in \hat{\mathcal{R}} \text{ such that } B(a, 1/m) \subset U\right\}.$$

The family

$$\mathcal{B} := \{B(a, 1/m)\colon (a, m) \in A \times \mathbb{N}\}$$

is a base of (X_1, ρ_1), i.e., for each $x \in X_1$ and an open set V in (X_1, ρ_1), if $x \in V$ then there exists $(a, m) \in A \times \mathbb{N}$ such that $x \in B(a, 1/m) \subset V$. As $\hat{\mathcal{R}}$ is an open cover of X_1 and $\mathcal{B}$ is a base of (X_1, ρ_1) we see that

$$\begin{gathered}\text{for each } x \in X_1 \text{ there exist } U \in \hat{\mathcal{R}} \text{ and } (a, m) \in A \times \mathbb{N} \\ \text{such that } x \in B(a, 1/m) \subset U.\end{gathered} \tag{3.4}$$

In particularly, $A^* \neq \emptyset$. By the axiom of choice and the definition of A^*, there exists a function $\varphi\colon A^* \to \hat{\mathcal{R}}$ such that for each $(a, m) \in A^*$ the inclusion $B(a, 1/m) \subset \varphi(a, m)$ holds. Since $A^* \subset A \times \mathbb{N}$, the set A^* is countable. Therefore $\varphi(A^*)$, which is a subfamily of $\hat{\mathcal{R}}$, is countable. We will show that the family $\varphi(A^*)$ covers X_1. In order to do this fix $x \in X_1$. By the property (3.4) there exist $U \in \hat{\mathcal{R}}$ and $(a, m) \in A \times \mathbb{N}$ such that $x \in B(a, 1/m) \subset U$. Hence, by the definition of A^* we claim that $(a, m) \in A^*$. Using the property of the function φ we get $x \in B(a, 1/m) \subset \varphi(a, m)$. Thus

$$X_1 = \bigcup_{(a,m) \in A^*} \varphi(a, m),$$

and so $\varphi(A^*)$ is a countable cover of X_1. Consequently, the family $\varphi(A^*) \setminus \{X_1 \setminus \mathrm{cl}(\Omega)\}$, which is a subfamily of $\mathcal{R}$, is a countable cover of $\mathrm{cl}(\Omega)$. This means that $\mathrm{cl}(\Omega)$ has a Lindelöf property in (X_1, ρ_1).

Now fix $f \in \mathcal{L}(\Omega)$. By Theorem 3.4, there exists a countable subset A of the set Ω^d such that $f|_{\Omega \setminus A}$ is continuous, which is a desired conclusion. □

Remark 3.6. Contrary to Theorem 3.1 we cannot replace continuity by uniform continuity in Theorems 3.4 and 3.5. Namely, set $(X_1, \rho_1) = (X_2, \rho_2) := \mathrm{E}(\mathbb{R})$, $\Omega := \mathbb{R}$ and $\mathbb{R} \ni x \mapsto f(x) := x^3$. Then $f \in \mathcal{L}(\Omega)$ as a continuous function. On the other hand $f|_{\Omega \setminus A}$ is not uniformly continuous on $\mathbb{R} \setminus A$ for each countable set $A \subset \Omega^d = \mathbb{R}$.

4. Applications

By Remark 1.2 there is a close relationship between continuous and limitable functions. A natural question arises how much are limitable and continuous functions different from each other? For any two functions $f_1\colon D\to Y$ and $f_2\colon D\to Y$ we define

$$A_{f_1,f_2} := \{x\in D\colon f_1(x)\neq f_2(x)\}.$$

In the proofs of Theorems 4.1 and 4.3 we use the following known fact on convergence of a sequence in an arbitrary metric space (X,ρ):

$$\begin{aligned}&\textit{for each } c\in X \textit{ and for each sequence } \mathbb{N}\ni n\mapsto y_n\in X\\ &\textit{the equality } \lim_{n\to\infty} y_n = c \textit{ holds if and only if for each increasing sequence}\\ &\mathbb{N}\ni k\mapsto n_k\in\mathbb{N} \textit{ there exists an increasing sequence } \mathbb{N}\ni l\mapsto k_l\in\mathbb{N}\\ &\textit{such that } \lim_{l\to\infty} y_{n_{k_l}} = c.\end{aligned} \tag{4.1}$$

Theorem 4.1. *If* $\mathrm{cl}(\Omega)$ *is compact in* (X_1,ρ_1) *and* $f\in\mathcal{L}(\Omega)$*, then there exists a uniformly continuous function* $f_0\colon\Omega\to X_2$ *such that one of the following two conditions holds:*

(i) *the set* A_{f,f_0} *is finite;*
(ii) *the set* A_{f,f_0} *is infinite and there exists a sequence* $\mathbb{N}\ni n\mapsto a_n\in\Omega$ *such that* $\{a_n\colon n\in\mathbb{N}\}=A_{f,f_0}$ *and*

$$\rho_2(f(a_n),f_0(a_n))\to 0 \quad \textit{as } n\to\infty. \tag{4.2}$$

Proof. Given $f\in\mathcal{L}(\Omega)$ set $f_0:=\hat f|_\Omega$, where $\hat f$ is defined by (1.2). First we prove that A_{f,f_0}, which is a subset of Ω, is countable. This is obvious if Ω is countable. Suppose now that Ω is not countable. By Remark 1.2, the function $\hat f$ is uniformly continuous, and so f_0 is uniformly continuous. The functions f and f_0 coincide at each continuity point of f. Indeed, if f is continuous at $x\in\Omega^d\cap\Omega$ (the case $x\in\Omega\setminus\Omega^d$ is evident) then

$$f(x)=\lim_{\Omega\ni t\to x} f(t)=\hat f(x)=f_0(x).$$

By Theorem 3.1(ii), there exists a countable set A such that f is continuous at each point x of $\Omega\setminus A$. Hence $A_{f,f_0}\subset A$, which means that A_{f,f_0} is countable.

Assume now that A_{f,f_0} is infinite and let $\mathbb{N}\ni n\mapsto a_n\in\Omega$ be a bijection of $\mathbb{N}$ onto A_{f,f_0}. We have to prove (4.2). To this end we apply (4.1) to the sequence $\mathbb{N}\ni n\mapsto b_n:=\rho_2(f(a_n),f_0(a_n))$. Fix an increasing sequence $\mathbb{N}\ni k\mapsto n_k\in\mathbb{N}$. By the compactness of $\mathrm{cl}(\Omega)$, there exists an increasing sequence $\mathbb{N}\ni l\mapsto k_l\in\mathbb{N}$ and an element $x\in\Omega^d$ such that $a_{n_{k_l}}\to x$ as $l\to\infty$. Since $f_0=\hat f$ on Ω we have $f_0(a_n)=\hat f(a_n)$ for $n\in\mathbb{N}$. By (1.2) and Remark 1.2 we see that

$$b_{n_{k_l}}=\rho_2\Big(f(a_{n_{k_l}}),f_0(a_{n_{k_l}})\Big)\to\rho_2(\hat f(x),\hat f(x))=0 \quad \text{as } l\to\infty.$$

Therefore, by (4.1), we have $b_n\to 0$ as $n\to\infty$, which proves (4.2). □

Remark 4.2. The compactness of $\mathrm{cl}(\Omega)$ cannot be omitted in Theorem 4.1. Indeed, set $(X_1,\rho_1)=(X_2,\rho_2):=\mathrm{E}(\mathbb{R})$, $\Omega:=\mathbb{R}$ and consider the function $f\colon\Omega\to\mathbb{R}$ given by

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

Obviously $f \in \mathcal{L}(\Omega)$. However, there is no a uniformly continuous function $f_0 \colon \Omega \to \mathbb{R}$ such that one of the conditions (i) or (ii) of Theorem 4.1 holds.

Theorem 4.3. *Let (X_2, ρ_2) be a complete metric space. If $f \colon \Omega \to X_2$ and there exists a uniformly continuous function $f_0 \colon \Omega \to X_2$ such that one of the conditions* (i) *or* (ii) *of* Theorem 4.1 *holds, then $f \in \mathcal{L}(\Omega)$.*

Proof. Fix f, f_0 and a sequence $\mathbb{N} \ni n \mapsto a_n \in \Omega$ as in the hypothesis of the theorem. If the condition (i) holds, then evidently $f \in \mathcal{L}(\Omega)$.

Assume now that the condition (ii) holds. Of course $f_0 \in \mathcal{L}(\Omega)$. By Remark 1.2, the function $\widehat{f_0} \colon \operatorname{cl}(\Omega) \to X_2$ is continuous. Fix $x \in \Omega^d$ and a sequence $\mathbb{N} \ni n \mapsto x_n \in \Omega$ such that

$$x_n \neq x \quad \text{for } n \in \mathbb{N} \qquad \text{and} \qquad \rho_1(x_n, x) \to 0 \quad \text{as } n \to \infty. \tag{4.3}$$

We claim that $f(x_n) \to \widehat{f_0}(x)$ as $n \to \infty$. In order to prove this we will apply (4.1) to the sequence $\mathbb{N} \ni n \mapsto f(x_n)$. Fix an increasing sequence $\mathbb{N} \ni k \mapsto n_k \in \mathbb{N}$. We consider two cases.

Case I: there exists $k_0 \in \mathbb{N}$ such that

$$x_{n_k} \in A_{f,f_0} \quad \text{for } k \geq k_0. \tag{4.4}$$

We define sequences $\mathbb{N} \ni l \to k_l \in \mathbb{N}$ and $\mathbb{N} \ni l \to m_l \in \mathbb{N}$ recursively as follows:

$$\begin{aligned} k_1 &:= \min\left(\{k \in \mathbb{N} \colon a_k = x_{n_{k_0}}\}\right), \\ m_1 &:= 1 + \max\left(\{j \in \mathbb{N} \colon j \geq k_0 \,\text{and}\, x_{n_j} \in A_{k_1}\}\right), \end{aligned} \tag{4.5}$$

where $A_p := \{a_k \colon k \in \mathbb{N} \,\text{and}\, 1 \leq k \leq p\}$ for $p \in \mathbb{N}$, and for each $l \in \mathbb{N}$,

$$\begin{aligned} k_{l+1} &:= \min(\{k \in \mathbb{N} \colon a_k = x_{n_{m_l}}\}), \\ m_{l+1} &:= 1 + \max\left(\{j \in \mathbb{N} \colon j \geq k_0 \,\text{and}\, x_{n_j} \in A_{k_{l+1}}\}\right). \end{aligned} \tag{4.6}$$

It follows from (4.3) that the sequence $\mathbb{N} \ni n \mapsto x_n$ does not contain any constant subsequence. This fact and the assumption $\{a_n \colon n \in \mathbb{N}\} = A_{f,f_0}$ (see hypothesis (ii) of the theorem) ensure that the sequences $\mathbb{N} \ni l \mapsto k_l \in \mathbb{N}$ and $\mathbb{N} \ni l \mapsto m_l \in \mathbb{N}$ are well defined. Moreover, by formulas (4.5) and (4.6), we have

$$x_{n_{m_l}} = a_{k_{l+1}} \quad \text{for } l \in \mathbb{N}, \tag{4.7}$$

and so $x_{n_{m_l}} \in A_{k_{l+1}} \setminus A_{k_l}$, which yields

$$k_l < k_{l+1} \quad \text{and} \quad m_l < m_{l+1} \quad \text{for } l \in \mathbb{N}. \tag{4.8}$$

Hence, by (4.8), the hypothesis (ii) and (4.3) we get

$$\begin{aligned}\rho_2(f(x_{n_{m_l}}), \widehat{f}_0(x)) &\le \rho_2(f(x_{n_{m_l}}), f_0(x_{n_{m_l}})) + \rho_2(f_0(x_{n_{m_l}}), \widehat{f}_0(x))\\ &= \rho_2(f(a_{k_{l+1}}), f_0(a_{k_{l+1}})) + \rho_2(f_0(x_{n_{m_l}}), \widehat{f}_0(x)) \to 0 \quad \text{as } l \to \infty.\end{aligned}$$

Consequently,

$$f\left(x_{n_{k_l}}\right) \to \widehat{f}_0(x) \quad \text{as } l \to \infty.$$

Case II: there does not exist $k_0 \in \mathbb{N}$ such that (4.4) holds. In this case there exists an increasing sequence $\mathbb{N} \ni l \mapsto k_l \in \mathbb{N}$ such that $x_{n_{k_l}} \in \Omega \setminus A_{f,f_0}$ for $l \in \mathbb{N}$. By (4.3), we get

$$f\left(x_{n_{k_l}}\right) = f_0\left(x_{n_{k_l}}\right) \to \widehat{f}_0(x) \quad \text{as } l \to \infty.$$

Combining the both cases and applying (4.1) we deduce that $f(x_n) \to \widehat{f}_0(x)$ as $n \to \infty$, and so $\lim_{t\to x} f(t) = \widehat{f}_0(x)$. This means $f \in \mathcal{L}(\Omega)$, which is the desired conclusion. □

The following corollary is an immediate consequence of Theorem 4.3.

Corollary 4.4. *Let (X_2, ρ_2) be a complete metric space. If $f\colon \Omega \to X_2$ and there exists a uniformly continuous function $f_0\colon \Omega \to X_2$ such that A_{f,f_0} is a non-empty countable set and*

$$\sum_{x\in A_{f,f_0}} \rho_2\left(f(x), f(x_0)\right) < +\infty,$$

then $f \in \mathcal{L}(\Omega)$.

Combining Theorems 4.1 and 4.3 we obtain the following result.

Theorem 4.5. *Let (X_2, ρ_2) be a complete metric space, the set* $\mathrm{cl}(\Omega)$ *be compact in (X_1, ρ_1) and let $f\colon \Omega \to X_2$. Then $f \in \mathcal{L}(\Omega)$ if and only if there exists a uniformly continuous function $f_0\colon \Omega \to X_2$ such that either the set A_{f,f_0} is finite or A_{f,f_0} is infinite and there exists a sequence $\mathbb{N} \ni n \mapsto a_n \in \Omega$ such that $\{a_n\colon n \in \mathbb{N}\} = A_{f,f_0}$ and $\rho_2(f(a_n), f_0(a_n)) \to 0$ as $n \to \infty$.*

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Institute of Mathematics and Computer Science
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129
PL-20-950 Lublin
Poland
E-mail: armen@kul.lublin.pl

Institute of Mathematics and Computer Science
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129
PL-20-950 Lublin
Poland
and
Institute of Mathematics and Information Technology
The State School of Higher Education in Chełm
Pocztowa 54
PL-22-100 Chełm
Poland
E-mail: partyka@kul.lublin.pl

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TWIERDZENIA TYPU ŁUZINA DLA FUNKCJI LIMESOWALNYCH

S t r e s z c z e n i e

W pracy wprowadzamy pojęcie funkcji limesowalnej (czyli takiej, która ma granicę w każdym punkcie skupienia jej dziedziny) i dowodzimy twierdzeń typu Łuzina dla takich funkcji. Podajemy również charakteryzację funkcji limesowalnych.

Słowa kluczowe: funkcja wartości granicznych, własność Lindelöfa, twierdzenie Łuzina