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2. Toute communications est présentée à la séance d'une Commission de la Société par un des members (avec deux opinions de spécialistes designés par la Rédaction). Elle doit lui être adressée directement par l'auteur.
3. L'article doit être écrit en anglais, français, allemand ou russe et débuté par un résumé en anglais ou en langue de la communication présentée. Dans tous les travaux écrits par des auteurs étrangers le titre et le résumé en polonais seront préparés par la rédaction. Il faut fournir le texte original qui ne peut contenir plus de 15 pages (plus 2 copies).
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TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

1. General information

The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

2. How to prepare a manuscript

To prepare the LaTeX 2e source file of your paper, copy the template file **instr.tex** with **Fig1.eps**, give the title of the paper, the authors with their affiliations/addresses, and go on with the body of the paper using all other means and commands of the standard class/style 'bull.cls'.

2.1. Example of a figure

Figures (including graphs and images) should be carefully prepared and submitted in electronic form (as separate files) in Encapsulated PostScript (EPS) format.

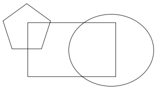

Fig. 1: The figure caption is located below the figure itself; it is automatically centered and should be typeset in small letters.

2.2. Example of a table

Tab. 1: The table caption is located above the table itself; it is automatically centered and should be typeset in small letters.

Description 1	Description 2	Description 3	Description 4
Row 1, Col 1	Row 1, Col 2	Row 1, Col 3	Row 1, Col 4
Row 2, Col 1	Row 2, Col 2	Row 2, Col 3	Row 2, Col 4

2.3. "Ghostwriting" and "guest authorship" are strictly forbiden

The printed version of an article is primary (comparing with the electronic version). Each contribution submitted is sent for evaluation to two independent referees before publishing.

3. How to submit a manuscript

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References

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Affiliation/Address

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Recherches sur les déformations no. 2

pp. 7–28

Aleksandr Bakhtin, Julian Ławrynowicz, Sergiy Plaksa, and Yuri Zelinskiĭ

LATE PROFESSOR PROMARZ TAMRAZOV (1933–2012) AND 20 YEARS OF SCIENTIFIC COOPERATION ŁÓDŹ-KYIV

Summary

Professor Promarz Tamrazov (*June 17, 1933), the outstanding scientist, our good friend, and member of the Editorial Board of *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.*, has passed away on February 11, 2012. We remember his research activities, obtained results and developed theories, solved open problems, as well as his engagement in initiation and prosperous development of scientific cooperation Łódź-Kyiv in mathemathics and physics since 1992 (Photo 1).

Keywords and phrases: obituary, Tamrazov, mathematics, physics, Polish-Ukrainian cooperation, cooperation Łódź-Kyiv

Photo 1. Professor Pomarz Tamrazov

Brief callendarium of the life

Professor Promarz Melikovich Tamrazov was a Corresponding Member of the National Academy of Sciences of Ukraine and a worldwide known mathematician for his works on complex and real analysis, geometric function theory, approximation theory, potential theory and combinatorics.

He was born on 17.06.1933 in Kiev. His mathematical faculties appeared very early. He told that when being a schoolboy of primary school, he played at home and skulked from time to time under a table, where his elder brother was solving mathematical school-problems under the supervision of parents. And when his elder brother was at a loss with the answer to the next problem, little boy Proma informed loudly a correct answer.

When Promarz studied in a high school, he became the winner of the Kiev municipal competition on mathematics among the schoolboys. In 1951 he finished the high school with the Gold Medal honor. From 1951 he studied at Kiev Polytechnic Institute and in 1956 graduated with honor in Mechanical and Heat Engineering, receiving an engineer degree (equivalent of the Master degree). Between 1956 and 1963 he worked in Kiev Polytechnic Institute and institutes of Ukrainian Academy of Sciences as engineer and assistant professor. In 1958–61 he undertook post-graduate studies in mathematics under the guidance of Professor V. A. Zmorovich. After proving a Ph.D./Candidate thesis in 1963 he had prepared in three years a brilliant Doctor Sciences thesis (Physics and Mathematics) that was proved on 29.01.1966 in Institute of Mathematics of Ukrainian Academy of Sciences, Kiev. Since 1963 he worked in the Institute of Mathematics of the Ukrainian Academy of Sciences as a research fellow and then as the head of a laboratory. He got the title of Professor in 1982. In 1989–2003 he was the head of the Department of Complex Analysis and Potential Theory and since 2003 the leading research fellow of the mentioned department. In 2006 P.M. Tamrazov was elected a Corresponding Member of the National Academy of Sciences of Ukraine. In addition, he took up the following collaboration positions: a lecturer (1970–71) and Professor (1988) at the Kiev State University, Professor at the Kiev Polytechnic Institute (1976–83), and Rector of the Peoples University of Modern Mathematics (1973–87).

Since 1958 he researched in mathematics and published 191 works (including a monograph *Smothnesses and Polynomial Approximation*, Naukova Dumka, Kiev, 1975). 13 Ph D and 4 Doctor of Sciences theses have been proved under his supervision.

He participated in many international conferences and congresses, and received the following grants:

- Long Term grant of ISF (International Science Foundation – Soros Foundation) – 1994, Principal Investigator;
- Grant of ICM-94 – 1994;
- Grant of ISF and Ukraine – 1995, Leader;

- Two grants of INTAS (European Union) – Call 1994 (1995–1998) and Call 1999 (2000–2003), Team Leader;
- Grant of the Royal Society (UK) – 1994;
- Grant of the Royal Swedish Academy of Sciences – 1997;
- Series of Soviet and Ukrainian grants;
- Grant of ICM-98 – 1998;
- Grant of NSF of USA for attending CMFT'01, 2001;
- Grant of ICM-02 – 2002;
- Travel Grant of International Mathematical Union for attending ICM-02 in Beijing, 2002.

He was a member of the ISAAC Board (1998–2002) and a member of the ISAAC Award Committee (1999).

Research activities of P. M. Tamrazov – Main topics

We list quite impressive variety of main research activities of Professor Tamrazov (20 items in our classification) ordered in four groups:

1. Univalent functions, conformal mappings, geometric function theory:
   - extremal problems, boundary behaviour, mappings of multiply and infinitely connected domains, problems on boundedness of functionals in noncompact classes of mappings, exact estimates of functionals;
   - extremal length, extremal metrics and quadratic differentials, general properties and applications to problems of geometric function theory and potential theory;
   - difference and differential contour-solid problems for holomorphic (and meromorphic) functions in the complex plane and in complex analytic spaces;
   - applications of geometric function theory to approximation theory;
   - extremal metric and modulus problems on nonorientable and twisted Riemannian manifolds.
2. Constructive function theory:
   - inverse and direct theorems of polynomial (and rational) approximation of functions on compacts of the complex plane, constructive characterization of functions;
   - complex finite-difference smoothnesses (of orders $k > 1$) for functions on sets of the complex plane;
   - problems of constructive function theory for complex finite-difference smothnesses of any order;
   - applications to the theory of singular integral operators;

- (strongly) local theory of approximation and (strongly) local constructive characterization of functions;
- finite-difference smothnesses of Cauchy integral operator and related singular operators;
- finite-difference smoothnesses of functions' superpositions and their application.

3. Potential theory:
   - equilibrium potentials of general condensers, their complete description;
   - Gonchar's extremal problem on capacities of condensers;
   - capacities and method of mixing of signed measures (charges);
   - removable singularities, subharmonic and plurisubharmonic extensions of functions in Euclidean, complex analytic and topological vector spaces;
   - finely subharmonic (and finely holomorphic) functions in the contour-solid and cluster problems;
   - Eremenko's extremal problem on harmonic functions.
4. Combinatorics:
   - harmonic analysis in vector spaces of complex finite and divided differences;
   - harmonic analysis for complex finite and divided differences of functions' superpositions.

Obtained results and developed theories

Professor Tamrazov proved the boundedness of certain functionals on noncompact classes of pairs of conformal mappings of doubly and multiply connected domains, and he proved the convergence of the analogues of the Blaschke product in such domains.

He investigated general properties of extremal lengths and extremal metrics:

- suggested an approach based on definitions in which volume (in the plane-areal) integrals are taken in the Lebesgue sense while linear integrals are taken in the lower Darboux sense;
- this allowed to take into consideration all metrics L-measurable in space (for the plane-areal) sense and all curves without requirement of local rectifiability; – this approach results in advantages from a general point of view and in applications;
- he established the local extremal property of extremal metrics;
- introduced a general limit modulus problem and proved the uniqueness of extremal metric for this problem;
- gave applications of these general results to extremal conformal mappings of multiply (infinitely) connected domains.

Photo 2: Professor Fyodor Kravchenko in front of the Kamenec Podilski castle (1987).

Photo 3: Profs. Julian Ławrynowicz and Yuri Trokhimchuk during a conference in Płock (1993).

Promarz Tamrazov solved extremal problems for conformal mappings associated with multipole quadratic differentials and gave complements to general coefficients theorem of J. A. Jenkins. He solved problems concerned with finding extremal metrics and moduli of some nonorientable and other twisted Riemannian manifolds, including the problem for Mobius strip tried by Pu in 1952 but not solved then.

Proma solved geometric problems related to conformal mappings, their boundary properties, symmetrization of multiply connected domains etc. He extended to an arbitrary bounded continuum inverse theorems of polynomial approximation in the complex plane known before only for some good piecewise smooth Jordan domains. He proved such theorems for a wide class of compacts.

Promarz established solid inverse theorems of polynomial approximation for the same continua and compacts (this problem was open even for the unit disc). He proved direct theorems of polynomial approximation and obtained constructive characterization of functions on some new classes of sets. He solved the open problem of local constructive characterization of functions; as a particular case this enabled to localize also the Jackson-Bernstein characterization of periodic functions on the real axis which was an open problem.

Proma solved the definition problem for difference smoothnesses of order $k > 1$ for functions on arbitrary sets of the complex plane, which was open even for the case when $k = 2$ and a set is a good piecewise smooth Jordan domain. He developed the theory of complex finite-difference smoothnesses of any order on general sets in the complex plane.

Professor Tamrazov established direct and inverse theorems of polynomial approximation, constructive characterization of functions. He proved theorems on finite-difference smoothnesses of conjugate harmonic functions, of Cauchy type integrals and related singular operators.

Promarz Tamrazov solved the problem of moduli of smoothness for functions' superpositions on sets of the complex plane which for a long time was open even for the case when simple functions on a real intervals and classical real moduli of smoothness were considered. He solved the difference contour-solid problems for holomorphic functions posed by Sewell in 1942, and developed a general contour-solid theory for holomorphic functions in open sets of the complex plane and in complex analytic spaces.

Promarz extended some of these results to meromorphic and subharmonic functions, holomorphic functions and mappings in complex analytic spaces and to moduli of smoothness of orders $k > 1$. He solved the differential boundary problem for analytic functions in arbitrary open set of the complex plane and at every fixed boundary point. This gave the ultimate positive solution for the problem discussed at an informal problem seminar in Zurich held in 1994 by participants of the International Congress of Mathematicians.

Proma solved the Gonchar's extremal problem on capacities of condensers and for this purpose he developed a new method based on mixing signed measures (charges). He extended the Brelot-Cartan Theorem on removable singularities for subharmonic functions to arbitrary polar sets (1983) and onto arbitrary sets of inner capacity zero (1993).

Promarz obtained characterization of sets removable under subharmonic extension of functions: first for singular capacitable sets (1983), and later on – without the restriction of capacitability (1993). He obtained results on plurisubharmonic extension of functions in complex analytic and topological vector spaces. In particular, he proved theorems on removability of perturbatively restricted singularities for plurisubharmonic functions in topological complex vector spaces. Such a theorem was announced in the 60th but not proved (in infinitely-dimensional spaces).

Proma established contour-solid and cluster theorems for finely subharmonic and finely holomorphic functions in finely open sets. He developed harmonic analysis in vector spaces of complex finite and divided differences.

Professor Tamrazov developed harmonic analysis for complex finite and divided differences of functions superpositions which was an open problem even for simple functions on real interval and classical real finite differences. These theories were needed for various applications in constructive and geometrical function theory. He developed methods for solving extremal problems associated with quadratic differentials having free poles.

Photo 4: Profs. Tamrazov (front row) and Ławrynowicz (back row, centre), and Dr. Przemysław Skibiński (right to Ł.) during a conference on complex analysis at Druzhba near Varna (1987).

Photo 5: Professors Tamrazov and Ławrynowicz during the VIIth ICAF at Kozubnik (1979).

Photo 6: Professors Trokhimchuk and Zielinskiĭ during the VIIth ICAF at Kozubnik (1979).

Solved open problems

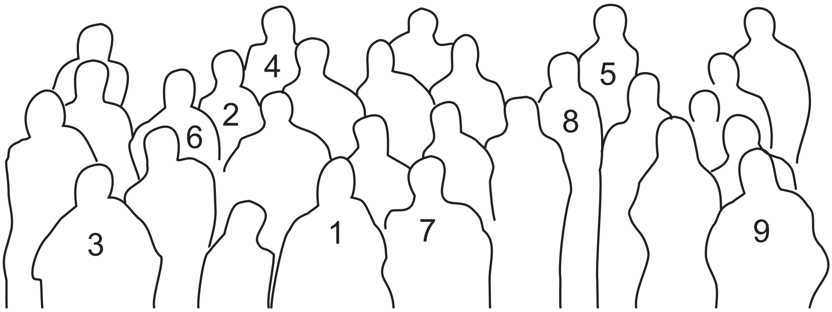
In addition we list open problems solved by Professor Tamrazov (in chronological order) which were posed and tried by other scientists:

1. The problem on boundedness of certain functionals on noncompact classes of pairs of conformal mappings of doubly (and multiply) connected domains and on convergence of analogues of the Blaschke products in doubly and multiply connected domains needed for extending of the theory of special classes of functions to multiply connected domains; this problem was posed by V. A. Zmorovich and discussed by A. I. Markushevich, S. Ya. Khavinson and H. Ts. Tumarkin. Proved the convergence of the mentioned analogues of the Blaschke product.
2. The problem on behaviour of conformal modulus of multiply connected domains under symmetrization, posed by I. P. Mitjuk.
3. The problem on conformal mapping of strip domains posed by A. A. Goldberg and applied by him in the theory of meromorphic functions.
4. The problem on boundary behaviour of conformal mapping posed by G. D. Suvorov in connection with compactification problems.
5. A uniqueness problem for an extremal problem of conformal mapping investigated by P. Duren.
6. A geometric function theory problem on boundedness of harmonization functionals posed by N. A. Lebedev at the International Congress of Mathematicians in 1966 and needed for solving open problems of approximation theory.
7. The solid problem of polynomial approximation of analytic functions which was open even for the unit disc (this latter particular problem was posed by V. K. Dzjadyk).
8. The contour-solid problem for analytic functions posed by Sewell in 1942. Obtained results enabled to solve a number of open problems of the theory of analytic and harmonic functions (about smoothnesses of conjugated harmonic functions and conformal mappings, of singular integral operators and solutions to singular integral equations), of approximation theory (in direct and inverse problems of polynomial approximation on the complex sets), of the theory of holomorphic functions and mappings of several complex variables etc.
9. The problem of defining finite-difference smoothnesses of functions in complex domains attacked by specialists during a long time. The founded approach enabled to extend to finite-difference smoothnesses of orders $k > 1$ various results of complex constructive theory known for $k = 1$.
10. The problem of finite differences and modules of smoothness for function superpositions, which for a long time was open even in simplest situations. The solving of this problem enabled to solve also a problem on finite-difference smoothnesses of conformal mapping.

Photo 7: Professors Natalia Zoriĭ and Julian Ławrynowicz (extreme left) during the XIth ICAF in Warsaw and Rynia (1994).

Photo 8: Prof. Zoriĭ during the Seminar in Częstochowa (2001).

Photo 9: Professors Tamrazov and Zoriĭ during the Summer School 1989 in Complex Analysis and Potential Theory at Kaciveli (Crimea).



Ad Photo 10 (next page)

Professor Promarz Tanrazov (1) and his colleagues, professors and doctors from Łódź (2–9), among a group of participants of the Seminar at Bd (2000) in honour of Professors Charzyński and Tietz:

2. Dorota Klim
3. Julian Ławrynowicz
4. Leon Mikołajczyk
5. Andrzej Nowakowski
6. Adam Paszkiewicz
7. Stanisław Romanowski
8. Andrzej Sukiennicki
9. Kazimierz Włodarczyk

11. The problem of local constructive characterization of functions. The developed approach enabled also to localize well known Jackson-Bernstein constructive characterization of periodic functions on the real axis (this latter particular localization problem was discussed by specialists for a long time, but was treated in unsuccessful ways not giving positive result).
12. The Gonchar's extremal problem on capacities of condensers.
13. The problem on removing of perturbatively restricted singularities for plurisubharmonic functions in complex vector topological spaces; tried, but not solved in the 60th.
14. The problem on the conformal modulus and extremal metric of a Riemannian Möbius strip posed by Pu in 1952.
15. The extremal problem on harmonic functions posed by A. Eremenko in 1995.
16. A parameterization for extremals of the Tchebotaryov's problem posed in 1925 and the problem formulated by H. Grötzsch in 1930 as the hyperbolic analog of the Tchebotaryov's problem has been established.

Promarz M. Tamrazov has died on 11.02.2012 after a prolonged illness.

Promarz Melikovich Tamrazov was a remarkable man: he was kind, responsive and exceptionally attentive to the people. Yet, mathematics was his true love. It was a sense and a happiness with all his life. Contacting with him at mathematical discussions, we were convinced repeatedly of his large mathematical talent. Sometimes the impression was created that there were practically no mathematical difficulties for him. He had tremendous mathematical intuition and shared generously ideas with disciples. We had a good luck to meet this eminent person on our creative way. He will remain in our memory as an intellectually gifted and outstanding person for ever.

Photo 10: Professor Promarz Tamrazov and his eight colleagues from Łódź (for their names see the proceding page) during the Seminar at Bd (2000).

Photo 11: Profs. Oleg Gerus and Julian Ławrynowicz (right-hand side) at the bank of Teterev river near Zhytomyr (2003).

Photo 12: A group of participants of the Bd Seminar 2005, including Dr. Oksana Sumyk (front row, centre) and Profs. Yaroslav G. Prytula, Zelinskiĭ and Ławrynowicz (back row, right-hand side).

Cooperation Łódź-Kyiv in mathematics and physics

Direct scientific cooperation between the

- Univeristy of Łódź
- Łódź Society of Sciences and Arts
- National Academy of Sciences of Ukraine, Institute of Mathematics
- National Academy of Sciences of Ukraine, Institute of Physics

was initiated officially in 1992, but regular cooperation between mathematicians and physicists of Łódź and Kyiv started much earlier. In particular, the coordinators from both sides, Professors Julian Ławrynowicz and Promarz Tamrazov had met for the first time in 1964 in Moscow. Personal contacts between Profs. Ławrynowicz, Fyodor G. Kravchenko and Yuri Trokhimchuk started in 1972 in Kamenec Podilski (Photos 2 and 3).

Already before 1992 exchange of scientific visits as well as participation in conferences and summer schools in Poland and Ukraine led to the following important contributions, in particular in relation with the IVth–Xth International Conferences on Analytic Functions (hareafter abbr. ICAF) held in Łódź (1966), Lublin (1970), Kraków (1974), at Kozubnik (1979), at Błażejewko (1982), in Lublin (1986), and at Szczyrk (1990), respectively:

Gutlyanski [Gu1, 2]
Karupu [Ka]
Ławrynowicz and Rembieliński [ŁR]
Melnichenko [Me]
Navoyan and Tamrazov [NT]
Pałka and Skibiński [PS] (Photo 4)
Tamrazov [T1–5] (Photos 4 and 5)
Trokhimchuk [Tr1, 2]
Zelinskiĭ [Ze1–3] (Photo 6)
Zoriĭ [Zo1] (Photos 7–9)

The leading topic in the cooperation agreement was *Potential theory vs. investigations on semiconductors and crystals.* In the decade 1992–2001 Łódź scientists took an artive part in several summer schools on complex analysis and potential theory in Crimea, in particular in the Hydrodynamical Institute of the National Academy of Sciences of Ukraine at Kaciveli, and in the International Conference on Complex Analysis and Potential Theory in Kyiv in 2001 (Figs. 1 and 2).

On the other hand, the Ukrainian scientists from Kyiv, Lviv, and Zhytomyr took an active part in the following events in Poland:

- XIth ICAF in Warszawa and at Rynia 1994
- XIIth ICAF in Lublin 1998
- Advanced Seminar on Deformations of Mathematical Structures Applied in Physics, Łódź and Warszawa 1992/97
- Workshop on Generalizations of Complex Analysis, Warszawa 1994
- The Finnish-Polish-Ukrainian Summer School in Complex Analysis, Lublin 1996

- Seminar (hereafter abbr. S): Geometrical Methods of Generalized Quaternionic Analysis with Applications in Physics, Będlewo (hereafter abbr. Bd) 1999
- S: Applied Algebraic Functions and Eigenfunctions (in Honour of Professors Charzyński and Tietz), Bd 2000 (Photo 10)
- S: Generalized Cauchy-Riemann Structures and Surface Properties of Crystals, Bd and Częstochowa 2001

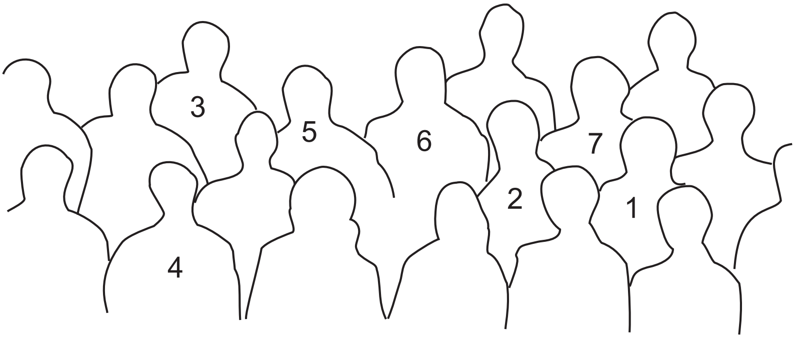
These activities led to the following important contributions:

Aguilar, Contreras, Cárdenas, Rutkowski, Gnatenko, and Bukivsky [AC]
Allyev [Al]
Cárdenas, Contreras, Gnatenko, and Rutkowski [CC]
Castillo, Contreras, Ławrynowicz, and Wojtczak [Ca]
Gaveau, Ławrynowicz, and Wojtczak [GŁ]
Gerus [Ge]
Gnatenko, Shigilchev, Rutkowski, Cárdenas, and Contreras [GS1]
—, —, —, Contreras, and Cárdenas [GS2]
Golberg [Go1]
Jakubowski and Zyskowska [JZ]
Kalynets and Kondratyuk [KK]
Kravchenko V. V. and Shapiro [KrS]
Krawiecki, Sukiennicki, and Wojtczak [KrSu] (Photo 10)
Ławrynowicz [Ł]
—, Martio, and Tamrazov [ŁM1, 2]
—, Porter, Ramírez, and Rembieliński [ŁP]
—, Rembieliński, and Succi [ŁRS]
— and Tamrazov [ŁT]
—, Wojtczak, Koshi, and Suzuki [ŁW]
Malinowski, Rembieliński, Tybor, and Papaloucas [MR]
Muliava and Sheremeta [MS]
Okhrimenko [O]
Pokhilevich [Po]
Poreda and Wilczyński [PoW]
Prytula Ya. Ya. [Pry]
Romanowski, Pietrzak, and Baldomir [RoP] (Photo 10)
Sheremeta [S]
Tamrazov [T6, 7]
—, Vuorinen, and Wielgus [TV]
Trokhimchuk [T3]
Urbaniak-Kucharczyk [U]
Wagner-Bojakowska [W]
Wojtczak and Gärtner [WG]
—, Urbaniak-Kucharczyk, Zasada, and Rutkowski [WU]
Zabolotskiĭ [Z]
Zoriĭ [Zo2–5]

In the decade 2002–2011 Łódź scientists took an active part in further meetings in Kyiv, Zhytomyr, and Lviv. In particular, since 2001 Julian Ławrynowicz is an elected foreign member of the Polish Scientific Society of Zhytomyr (Photo 11). The Society is not restricted to local friends of science of Polish origin, but has as members also Ukrainians interested in the Polish culture.

On the other hand, the Ukrainian scientists related to Kyiv, Lviv, and Zhytomyr took an active part in the following events in Poland:

- XIIIth ICAF at Będlewo (abbr. Bd) 2002
- XIVth ICAF in Chełm 2007



Ad Photo 13 (next page)

Professor Yuri Zelinskiĭ (1) with his Ukrainian colleagues, professors (2–4), and colleagues, professors and doctors from Łódź (5–7), among a group of participants of the XIIIth ICAF at Bd:

2. Oleg Gerus
3. Anatoly Golberg
4. Natalia Zoriĭ
5. Piotr Liczberski
6. Wiesław Majchrzak
7. Krystyna Skalska

- XVth ICAF in Chełm 2011
- International Conference: Ideas of Albert Abraham Michelson in Mathematical Physics, Bd 2002
- Seminar (abbr. S): Generalized Cauchy-Riemann Structures, Complex Approximation, and Surface Properties of Crystals, Bd 2003
- S: Applied Quaternionic and Finslerian Structures, Bd 2004
- S: Lvov Mathmatical School in the Period 1915–45 as Seen Today, Bd 2005 (Photo 12)
- S: Applied Complex and Quaternionic Approximation vs. Finslerian Structures, Bd 2006
- S: (Hyper)Complex Methods, Chaotic Features, Fractals, and Physical Applications, Bd 2007
- Hypercomplex Seminar (abbr. HS) 2008: Foliation Modelling of Hypercomplex Crystal Geometry vs. Randers-Ingarden and Fractal Structures, and Nanostructures, Bd 2008
- HS 2009: From Schauder Basis to Hypercomplex, Randers-Ingarden and Fractal Structures, and Nanostructures, Bd 2009
- HS 2010: (Hyper)Complex and Randers-Ingarden Structures in Mathematics and Physics, Bd 2010
- HS 2011: (Hyper)Complex Function Theory, Dolbeault Cohomology, Fractals, and Physics, Bd 2011
- HS 2012: (Hyper)Complex Function Theory, Regression, (Crystal) Lattices, Fractals, Chaos, and Physics; dedicated to the memory of Professor Promarz Tamrazov on the occasion of 20 years of the direct cooperation agreement Łódź-Kyiv, Bd 2012

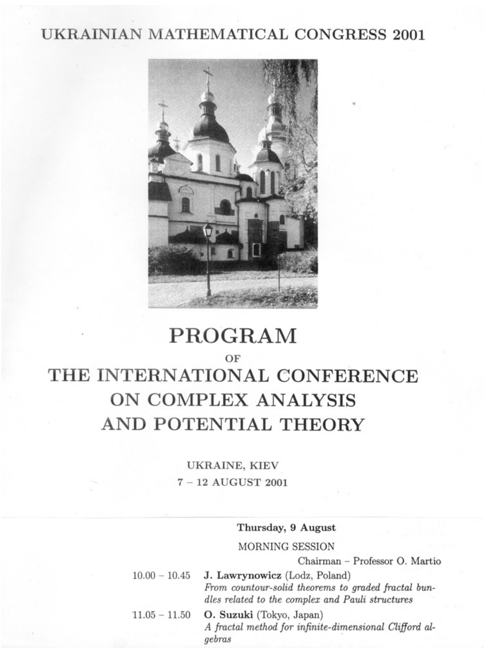
Photo 13: Professor Yuri Zelinskiĭ, his three colleagues from Ukraine, and three colleagues from Łódź (for their names see the proceding page) during the XIIIth ICAF at Bd (2002).

Photo 14: XIIIth ICAF at Bd (2002) – Prof. Zelinskiĭ during a free discussion meeting.

Photo 15: Bd Seminar 2007 – Profs. Zoriĭ and Ławrynowicz during a free discussion meeting

These activities led to the following important contributions:

Bojarski, Ławrynowicz, and Prytula Ya. G. [BŁ1, 2]
Casillo Pérez and Kravchenko V. [CaK]
Chuyko [Ch]
Denega [D]
Filevych and Sheremeta [FS]
Golberg [Go2] (Photo 13)
Ibáñez [I1–4]
Kharkovych and Zhyhallo [KhZ]
Kravchenko V. and Ramírez Tachiquin [KrR]
Ławrynowicz, Suzuki, and Castillo [ŁS]
Luna and Shapiro [LuS]
Mierzejeski [Mi1–6]
– and Shpakivskyi (Szpakowski) [MiS]
Moneta and Pantelica [MoP]
Prytula Ya. G. [Pr] (Photo 12)
Rembieliński and Smoliński [RS]
Sheremeta and Sumyk [SS]
Shpakivskyi (Szpakowski) [Sh]
— and Plaksa [ShP]
Slyusarchuk [Sl]
Wilczyński [Wi]
Zając, Kalchuk, and Stepanyk [ZaK]
Zasada, Wojtczak, and Surry [ZaW]
Zelinskiĭ [Ze3–6] (Photos 13, 14)
Zhyhallo [Zh]
Zoriĭ [Zo6–9] (Photos 13, 15)


UKRAINIAN MATHEMATICAL CONGRESS 2001

PROGRAM
OF
THE INTERNATIONAL CONFERENCE
ON COMPLEX ANALYSIS
AND POTENTIAL THEORY

UKRAINE, KIEV
7 – 12 AUGUST 2001

Thursday, 9 August
MORNING SESSION
Chairman – Professor O. Martio
10.00 – 10.45 **J. Lawrynowicz** (Lodz, Poland) *From countour-solid theorems to graded fractal bundles related to the complex and Pauli structures*
11.05 – 11.50 **O. Suzuki** (Tokyo, Japan) *A fractal method for infinite-dimensional Clifford algebras*

Fig. 1: The Kyiv Conference 2001.

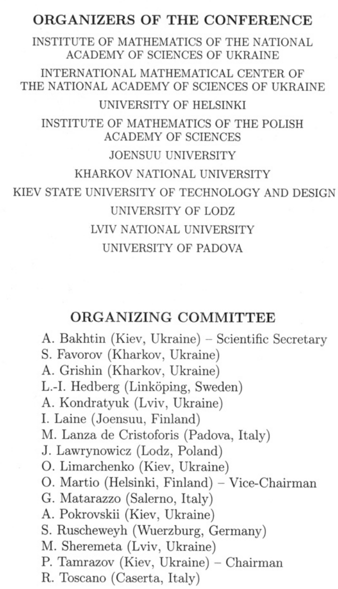

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Fig. 2: Organizers of the Kyiv Conference 2001.

Abbreviations (hereafter): BCP – Banach Center Publications; E. – Ed. by = Edited by; BS – Bull. Soc. Sci. Lettres Łódź = Bulletin de la Société des sciences et des Lettres de Łódź; JŁ – Julian Ławrynowicz; RD – Sér. Rech. Déform. = Série: Recherches sur les déformations.

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 29, 2012

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S t r e s z c z e n i e

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ON THE DIRICHLET PROBLEM FOR GENERAL DEGENERATE BELTRAMI EQUATIONS

Summary

We study the Dirichlet problem for general degenerate Beltrami equation $\overline{\partial} f = \mu\partial f + \nu\overline{\partial f}$ in the unit disk. New criteria for the existence of regular solutions are proven.

Keywords and phrases: Dirichlet problem, regular solutions, Beltrami equations with two characteristics

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. Throughout this paper we use the notations $B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$, $\mathbb{B}(r) := B(0, r)$, $\mathbb{B} := \mathbb{B}(1)$, and $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$.

The Beltrami equation of the first type

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1}$$

is basic in the theory of quasiconformal mappings in the plane. It is the equation that provides the connection of the geometric theory of quasiconformal mappings to complex analysis and elliptic PDEs. Here $f_{\overline{z}} = \overline{\partial} f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of $f = u + iv$ in the variables x and y, respectively, and $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e. For the equation (1) the existence problem was resolved for the uniformly elliptic case when $\|\mu\|_\infty < 1$, see e.g. [1, 2, 21]. The existence problem for degenerate Beltrami equation (1) when

$$K_\mu(z) \ : \ = \ \frac{1+|\mu(z)|}{1-|\mu(z)|} \ \notin \ L^\infty \tag{2}$$

is currently an active area of research, see e.g. the monographs [14] and [22] and the surveys [13] and [29] and further references therein.

On the other hand, the Beltrami equations of the second type

$$f_{\overline{z}} \ = \ \nu(z) \cdot \overline{f_z} \tag{3}$$

play a great role in many problems of mathematical physics, see e.g. [19]. In this connection, we established a series of criteria on existence of regular solutions for the Beltrami equations with two characteristics

$$f_{\overline{z}} \ = \ \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z} \tag{4}$$

in our recent papers [4]– [6]. There we called a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(D)$ by a *regular solution* of (4) if f satisfies (4) a.e. and $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e.

Boundary value problems for the Beltrami equations are due to the well-known Riemann dissertation in the case of $\mu(z) \equiv 0$ and $\nu(z) \equiv 0$ and to the papers of Hilbert (1904, 1924) and Poincare (1910) for the corresponding Cauchy–Riemann system. The Dirichlet problem was well studied for uniformly elliptic systems and for the corresponding Beltrami equations (4) when $K_{\mu,\nu} \in L^\infty$, see e.g. [2] and [31].

Recall that every analytic function f in a domain D in $\mathbb{C}$ satisfies the simplest Beltrami equation $f_{\bar{z}} = 0$ with $\mu(z) \equiv 0$ and $\nu(z) \equiv 0$. If an analytic function f given in the unit disk is continuous in its closure, then by the Schwarz formula

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \operatorname{Re} f(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}\,, \tag{5}$$

see, e.g., Section 8, Chapter III, Part 3 in [16]. Thus, the analytic function f in the unit disk $\mathbb{B}$ is defined, up to a purely imaginary additive constant ic, $c = \operatorname{Im} f(0)$, by its real part $\varphi(\zeta) = \operatorname{Re} f(\zeta)$ on the boundary of $\mathbb{B}$.

The Dirichlet problem for the Beltrami equation (4) in a domain $D \subset \mathbb{C}$ is the problem on the existence of a continuous function $f : D \to \mathbb{C}$ having partial derivatives of the first order a.e., satisfying (4) a.e. and such that

$$\lim_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \qquad \forall\ \zeta \in \partial D \tag{6}$$

for a prescribed continuous function $\varphi : \partial D \to \mathbb{R}$.

If $\varphi(\zeta) \not\equiv \text{const}$, then a *regular solution* of the Dirichlet problem

$$\begin{cases} f_{\overline{z}} \ = \ \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z}, \ \ z \in D, \\ \lim\limits_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \qquad \forall\ \zeta \in \partial D \end{cases} \tag{7}$$

in a domain D is a continuous, discrete and open mapping $f : D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ with its Jacobian $J_f(z) \neq 0$ a.e. satisfying (4) a.e. and the condition (6).

Recall that a mapping $f : D \to \mathbb{C}$ is called *discrete* if the preimage $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and *open* if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$.

The Dirichlet problem for the degenerate Beltrami equations of the first type (1) in the unit disk was recently studied in [10]. Moreover, in [20] it was proven a series of new criteria in terms of K_μ on the existence of regular solutions of the Dirichlet problem for the degenerate Beltrami equation (1) with continuous boundary data in an arbitrary Jordan domain.

The purpose of this paper is to study the Dirichlet problem in the unit disk with continuous boundary data for degenerate Beltrami equation of the form (4) with measurable coefficients $\mu(z)$, $\nu(z)$, satisfying the inequality $|\mu(z)| + |\nu(z)| < 1$ a.e. The degeneracy of the ellipticity is controlled by the dilatation coefficient

$$K_{\mu,\nu}(z) \;:=\; \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|} \in L^1_{\mathrm{loc}}. \tag{8}$$

To solve the Dirichlet problem with continuous boundary data we impose on $K_{\mu,\nu}(z)$ some additional conditions and give new criteria for the existence of regular solutions.

2. Preliminaries

To derive criteria for existence of regular solutions for the Dirichlet problem (7) we make use of the approximate procedure based on the existence theorems for the case $K_{\mu,\nu} \in L^\infty$ given in [2] and convergence theorems for the Beltrami equations (4) when $K_{\mu,\nu} \in L^1_{\mathrm{loc}}$ established in [5]. The Arzela-Askoli theorem combined with moduli techniques is also used.

In particular, by Theorems 5.1 and 6.1 and the point 8.1 in [2], see also Theorem VI.2.2 and the point VI.2.3 in [21] and (5) above, we have the following statement.

Proposition 2.1. *Let $\varphi : \partial\mathbb{B} \to \mathbb{R}$ be a nonconstant continuous function and $K_{\mu,\nu} \in L^\infty$ in the unit disk $\mathbb{B}$. Then the Dirichlet problem (7) in the unit disk $\mathbb{B}$ has a unique regular solution f normalized by* $\mathrm{Im} f(0) = 0$. *Moreover, this solution has the representation*

$$f = \mathcal{A} \circ g \tag{9}$$

where $g : \overline{\mathbb{B}} \to \overline{\mathbb{B}}$ is a homeomorphic regular solution of the equation

$$g_{\overline{z}} \;=\; \mu(z) \cdot g_z + \frac{\overline{\mathcal{A}'(g(z))}}{\mathcal{A}'(g(z))} \cdot \nu(z) \cdot \overline{g_z} \tag{10}$$

in $\mathbb{B}$ normalized by $g(0) = 0$, $g(1) = 1$, and $\mathcal{A} : \mathbb{B} \to \mathbb{C}$ is an analytic function such that

$$\mathcal{A}(w) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(g^{-1}(\zeta)) \cdot \frac{\zeta + w}{\zeta - w} \frac{d\zeta}{\zeta}\ . \tag{11}$$

Remark 2.1. Let $\mu^* : \mathbb{C} \to \mathbb{C}$ be the function that coincide a.e in $\mathbb{B}$ with

$$\frac{g_{\overline{z}}}{g_z} = \mu(z) + \nu(z) \cdot \frac{\overline{g_z}}{g_z} \cdot \frac{\overline{\mathcal{A}'(g(z))}}{\mathcal{A}'(g(z))} \tag{12}$$

and is equal to 0 outside of $\mathbb{B}$. Then $K_{\mu^*} \leq K_{\mu,\nu}$ a.e. in $\mathbb{B}$ and there is a regular solution $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the equation $G_{\overline{z}} = \mu^* G_z$ such that $G(0) = 0$, $|G(1)| = 1$ and $G(\infty) = \infty$. Moreover, $G = h \circ g$ in $\overline{\mathbb{B}}$ where $h : \mathbb{B} \to G(\mathbb{B})$ is a conformal mapping with $h(0) = 0$ and $h'(0) > 0$. Thus,

$$f = \mathcal{A} \circ h^{-1} \circ G \tag{13}$$

and

$$\mathcal{A}(w) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(G^{-1}(h(\zeta))) \cdot \frac{\zeta + w}{\zeta - w} \frac{d\zeta}{\zeta} \ . \tag{14}$$

Denote such f, g, $\mathcal{A}$, G and h by $f_{\mu,\nu,\varphi}$, $g_{\mu,\nu,\varphi}$, $\mathcal{A}_{\mu,\nu,\varphi}$ $G_{\mu,\nu,\varphi}$ and $h_{\mu,\nu,\varphi}$, respectively.

Recall also that, given a family of paths Γ in $\mathbb{C}$, a Borel function $\rho : \mathbb{C} \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \rho(z)\,|dz| \ \geq \ 1 \tag{15}$$

for each $\gamma \in \Gamma$. The *modulus* of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\ dxdy \ . \tag{16}$$

Remark 2.2. Note the following inequality for a quasiconformal mapping $f : D \to \mathbb{C}$, see e.g. V(6.6) in [21],

$$M(f(\Gamma)) \ \leq \ \int\limits_{\mathbb{C}} K(z) \cdot \rho^2(z)\ dxdy. \tag{17}$$

This inequality holds for every path family Γ in D and for all $\rho \in adm\,\Gamma$ where

$$K(z) \ = \ \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \tag{18}$$

is the (local) maximal dilatation of the mapping f at a point $z \in D$.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Delta(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. Recall that a *ring domain*, or shortly a *ring* in $\overline{\mathbb{C}}$ is a domain R whose complement $\overline{\mathbb{C}} \setminus R$ consists of two connected components.

The following statement is a direct consequence of the known estimate of the capacity of a ring formulated in terms of moduli, see e.g. Lemma 2.16 in [5].

Lemma 2.1. *Let $f : D \to \mathbb{C}$ be a homeomorphism with $\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta > 0$ and let z_0 be a point in D, $\zeta \in B(z_0, r_0)$, $r_0 < dist\,(z_0, \partial D)$. Then*

$$s(f(\zeta), f(z_0)) \ \leq \ \frac{32}{\Delta} \cdot exp\left(-\frac{2\pi}{M(\Delta(fC, fC_0, fA))}\right) \tag{19}$$

where $C_0 = \{z \in \mathbb{C} : |z - z_0| = r_0\}$, $C = \{z \in \mathbb{C} : |z - z_0| = |\zeta - z_0|\}$ and $A = \{z \in \mathbb{C} : |\zeta - z_0| < |z - z_0| < r_0\}$.

Recall that, for points $z, \zeta \in \overline{\mathbb{C}}$, the *spherical (chordal) distance* $s(z, \zeta)$ between z and ζ is given by

$$s(z,\zeta) \ = \ \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta \,, \tag{20}$$

$$s(z,\infty) \ = \ \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \,.$$

Here $\delta(A)$ denotes the spherical diameter of a set $A \subset \mathbb{C}$, i.e. $\sup\limits_{z,\zeta \in A} s(z, \zeta)$.

3. BMO, VMO and FMO functions

Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of *bounded mean oscillation* in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B| \, dm(z) < \infty \,, \tag{21}$$

where the supremum is taken over all discs B in D, $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$ and

$$u_B = \frac{1}{|B|} \int\limits_B u(z) \, dm(z) \,.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [18] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. [15] and [24].

A function φ in BMO is said to have *vanishing mean oscillation*, abbr. $\varphi \in \mathrm{VMO}$, if the supremum in (21) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [28]. There exists a number of papers devoted to the study of partial differential equations with coefficients of the class VMO.

Remark 3.1. Note that $W^{1,2}(D) \subset VMO\,(D)$, see e.g. [7].

Following [17], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \quad ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \ dxdy \ < \ \infty \tag{22}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = ⨍_{B(z_0,\varepsilon)} \varphi(z) \ dxdy$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon)$ with small $\varepsilon > 0$. We also say that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation* in D, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in FMO$, if (22) holds at every point $z_0 \in D$.

Remark 3.2. Clearly BMO $\subset$ FMO. There exist examples showing that FMO is not $\mathrm{BMO_{loc}}$, see e.g. [14]. By definition FMO $\subset L^1_{\mathrm{loc}}$ but FMO is not a subset of L^p_{loc} for any $p > 1$ in comparison with $\mathrm{BMO_{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$.

Proposition 3.1. *If, for some collection of numbers* $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}} \quad ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \ < \infty \,, \tag{23}$$

then φ *is of finite mean oscillation at* z_0.

Corollary 3.1. *If, for a point* $z_0 \in D$,

$$\overline{\lim_{\varepsilon\to 0}} \quad ⨍_{B(z_0,\varepsilon)} |\varphi(z)| \ dxdy \ < \ \infty \ , \tag{24}$$

then φ *has finite mean oscillation at* z_0.

Remark 3.3. Note that the function $\varphi(z) = log\frac{1}{|z|}$ belongs to BMO in the unit disk $\mathbb{B}$, see e.g. [24], p. 5, and hence also to FMO. However, $\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that the condition (24) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

Lemma 3.1. *Let* $\varphi : D \to \mathbb{R}$ *be a nonnegative function with finite mean oscillation at* $0 \in D$ *and let* φ *be integrable in* $B(0, e^{-1}) \subset D$. *Then*

$$\int\limits_{A(\varepsilon, e^{-1})} \frac{\varphi(z)\, dxdy}{\left(|z| \log \frac{1}{|z|}\right)^2} \leq \ C \cdot \log\log \frac{1}{\varepsilon} \qquad\qquad \forall\, \varepsilon \in (0, e^{-e}) \tag{25}$$

Here we use the notation $A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z| < \varepsilon_0\}$.

4. The main lemma

The following lemma is the main tool for obtaining criteria of the existence of regular solutions of the Dirichlet problem for the Beltrami equations with two characteristics in the unit disk.

Lemma 4.1. *Let μ, $\nu : \mathbb{B} \to \mathbb{C}$ be measurable functions with $K_{\mu,\nu} \in L^1(\mathbb{B})$. Suppose that for every $z_0 \in \overline{\mathbb{B}}$ there exist $\varepsilon_0 = \varepsilon(z_0)$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, such that*

$$0 \ < \ I_{z_0}(\varepsilon) \ : \ = \ \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t) \ dt \ < \ \infty \ , \tag{26}$$

and such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z) \cdot \psi^2_{z_0,\varepsilon}(|z-z_0|) \ dxdy \ = \ o(I^2_{z_0}(\varepsilon)) \tag{27}$$

as $\varepsilon \to 0$. Then the Beltrami equation (4) has a regular solution f satisfying the boundary condition (6) for each nonconstant continuous function $\varphi : \partial\mathbb{B} \to \mathbb{R}$.

Here we assume that μ and ν are extended by zero outside of the unit disk $\mathbb{B}$.

Proof. Setting

$$\mu_n(z) \ = \ \begin{cases} \mu(z) \ , & \text{if } K_{\mu,\nu}(z) \le n, \\ 0 \ , & \text{otherwise in } \mathbb{C}, \end{cases} \tag{28}$$

and

$$\nu_n(z) \ = \ \begin{cases} \nu(z) \ , & \text{if } K_{\mu,\nu}(z) \le n, \\ 0 \ , & \text{otherwise in } \mathbb{C}, \end{cases} \tag{29}$$

we have that $K_{\mu_n,\nu_n}(z) \le n$ in $\mathbb{C}$. Denote by f_n, g_n, $\mathcal{A}_n$, G_n and h_n the functions $f_{\mu_n,\nu_n,\varphi}$, $g_{\mu_n,\nu_n,\varphi}$, $\mathcal{A}_{\mu_n,\nu_n,\varphi}$ $G_{\mu_n,\nu_n,\varphi}$ and $h_{\mu_n,\nu_n,\varphi}$, respectively, from Proposition 2.1 and Remark 2.1.

Let Γ_ε be a family of all paths joining the circles $C_\varepsilon = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ and $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$ in the ring $A_\varepsilon = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$. Let also ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in (0,\infty)$. Such a function ψ^* exists by the theorem of Lusin, see e.g. [27], p. 69. Then the function

$$\rho_\varepsilon(z) = \begin{cases} \psi^*(|z-z_0|)/I_{z_0}(\varepsilon), & \text{if } z \in A_\varepsilon, \\ 0, & \text{if } z \in \mathbb{C}\backslash A_\varepsilon, \end{cases}$$

is admissible for Γ_ε. Hence by Remark 2.2 applied to G_n

$$M(G_n\Gamma_\varepsilon) \le \int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z) \cdot {\rho_\varepsilon}^2(|z-z_0|) \ dxdy \, ,$$

and, by the condition (27), $M(G_n\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to the parameter $n = 1, 2, \ldots$.

Thus, in view of the normalization $G_n(0) = 0$, $|G_n(1)| = 1$ and $G_n(\infty) = \infty$, the sequence G_n is equicontinuous in $\overline{\mathbb{C}}$ with respect to the spherical distance by Lemma 2.1 with $\Delta = 1/\sqrt{2}$. Consequently, by the Arzela–Ascoli theorem, see e.g. [8], p. 267, and [9], p. 382, it has a subsequence G_{n_l} which converges uniformly in $\overline{\mathbb{C}}$ with respect to the spherical metric to a continuous mapping G in $\overline{\mathbb{C}}$ with the normalization

$G(0) = 0$, $|G(1)| = 1$ and $G(\infty) = \infty$. $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism of the class $W^{1,1}_{\rm loc}(\mathbb{C})$ by Corollary 3.8 in [5].

Hence by the Rado theorem, see e.g. Theorem II.5.2 in [12], $h_{n_l} \to h$ as $l \to \infty$ uniformly in $\overline{\mathbb{B}}$ where $h : \overline{\mathbb{B}} \to G(\overline{\mathbb{B}})$ is the conformal mapping of $\mathbb{B}$ onto $G(\mathbb{B})$ with the normalization $h(0) = 0$ and $h'(0) > 0$. Moreover, since the locally uniform convergence $G_{n_l} \to G$ and $h_{n_l} \to h$ of the sequences G_{n_l} and h_{n_l} is equivalent to their continuous convergence, i.e., $G_{n_l}(z_l) \to G(z_*)$ if $z_l \to z_*$ and $h_{n_l}(\zeta_l) \to h(\zeta_*)$ if $\zeta_l \to \zeta_*$, see [Du], p. 268, and since G and h are injective, it follows that $G^{-1}_{n_l} \to G^{-1}$ and $h^{-1}_{n_l} \to h^{-1}$ continuously, and hence locally uniformly.

Then we have that $\mathcal{A}_{n_l} \to \mathcal{A}$ locally uniformly in $\mathbb{B}$ where

$$\mathcal{A}(w) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(G^{-1}(h(\zeta))) \cdot \frac{\zeta + w}{\zeta - w} \frac{d\zeta}{\zeta} \,. \tag{30}$$

Note that the analytic functions $\mathcal{A}_{n_l}$ and $\mathcal{A}$ are not constant and hence $\mathcal{A}'_{n_l}$ and $\mathcal{A}'$ have only isolated zeros in $\mathbb{B}$. Thus, by Theorem 3.1 and Corollary 3.8 in [5] $g_{n_l} \to g$ where $g = h^{-1} \circ G : \overline{\mathbb{B}} \to \overline{\mathbb{B}}$ is a homeomorphic $W^{1,1}_{\rm loc}$ solution in $\mathbb{B}$ of the quasilinear equation

$$g_{\overline{z}} \,=\, \mu(z) \cdot g_z + \frac{\overline{\mathcal{A}'(g(z))}}{\mathcal{A}'(g(z))} \cdot \nu(z) \cdot \overline{g_z} \tag{31}$$

with the normalization $g(0) = 0$ and $g(1) = 1$. Hence $f_{n_l} \to f$ where $f = \mathcal{A} \circ g$ is a continuous discrete open $W^{1,1}_{\rm loc}$ solution in $\mathbb{B}$ of (4).

Next, note that $\operatorname{Re} \mathcal{A}_{n_l} \to \operatorname{Re} \mathcal{A}$ uniformly in $\overline{\mathbb{B}}$ by the maximum principle for harmonic functions and $\operatorname{Re} \mathcal{A} = \varphi \circ g^{-1}$ on $\partial\mathbb{B}$ and, consequently, $\operatorname{Re} f_{n_l} \to \operatorname{Re} f$ uniformly in $\overline{\mathbb{B}}$ and $\operatorname{Re} f = \varphi$ on $\partial\mathbb{B}$, i.e., f is a continuous discrete open $W^{1,1}_{\rm loc}$ solution of the Dirichlet problem (6) in $\mathbb{B}$ to the equation (4). It remains to show that $J_f(z) \neq 0$ a.e. in $\mathbb{B}$.

By a change of variables which is permitted because g_{n_l} and $\tilde{g}_{n_l} = g^{-1}_{n_l}$ belong to the class $W^{1,2}_{\rm loc}$, see e.g. Lemmas III.2.1 and III.3.2 and Theorems III.3.1 and III.6.1 in [21], we obtain that for large enough l

$$\int\limits_{B} |\partial \tilde{g}_{n_l}|^2 \; dudv \;\leq \int\limits_{\tilde{g}_{n_l}(B)} \frac{dxdy}{1 - k_l(z)^2} \;\leq \int\limits_{B^*} K_{\mu,\nu}(z) \; dxdy \;<\; \infty \tag{32}$$

where $k_l(z) = |\mu_{n_l}(z)| + |\nu_{n_l}(z)|$ and B^* and B are relatively compact domains in $\mathbb{B}$ and $\tilde{g}(\mathbb{B})$, respectively, such that $\tilde{g}(\bar{B}) \subset B^*$. The relation (32) implies that the sequence $\tilde{g}_{n_l}$ is bounded in $\mathrm{W}^{1,2}(B)$, and hence $g^{-1} \in \mathrm{W}^{1,2}_{\rm loc}$, see e.g. Lemma III.3.5 in [25] or Theorem 4.6.1 in [11]. The latter condition brings in turn that g has $(N^{-1})-$property, see e.g. Theorem III.6.1 in [21], and hence $J_g(z) \neq 0$ a.e., see Theorem 1 in [23]. Thus, $f = \mathcal{A} \circ g$ is a regular solution of the Dirichlet problem (6) to the equation (4).

Corollary 4.1. *Let μ, ν : $\mathbb{B} \to \mathbb{C}$ be measurable functions with $K_{\mu,\nu} \in L^1(\mathbb{B})$. Suppose that for every $z_0 \in \overline{\mathbb{B}}$ there is $\varepsilon_0 > 0$ such that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z)\cdot\psi^2(|z-z_0|)\ dxdy\ \leq\ O\left(\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\ dt\right) \tag{33}$$

as $\varepsilon\to 0$, where $\psi:(0,\infty)\to(0,\infty)$ is a measurable function with

$$\int\limits_0^{\varepsilon_0}\psi(t)\ dt\ =\ \infty\ ,\qquad 0\ <\ \int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\ dt<\ \infty\qquad\forall\ \varepsilon\in(0,\varepsilon_0)\ . \tag{34}$$

Then the Beltrami equation (4) has a regular solution f satisfying the boundary condition (6) for each nonconstant continuous function $\varphi:\partial\mathbb{B}\to\mathbb{R}$.

5. Existence theorems

Everywhere further we assume that the functions μ and $\nu:\mathbb{B}\to\mathbb{B}$ are extended by zero outside of the unit disk $\mathbb{B}$ in $\mathbb{C}$.

Theorem 5.1. *Let μ and $\nu:\mathbb{B}\to\mathbb{B}$ be measurable functions such that*

$$K_{\mu,\nu}(z)\ =\ \frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|}\ \leq\ Q(z)\ \in\ FMO\ . \tag{35}$$

Then the Dirichlet problem (7) in the disk $\mathbb{B}$ has regular solution for each nonconstant continuous function $\varphi:\partial\mathbb{B}\to\mathbb{R}$.

Proof. Lemma 4.1 yields this conclusion by choosing

$$\psi_{z_0,\varepsilon}(t)\ =\ \frac{1}{t\log\frac{1}{t}}\ , \tag{36}$$

see also Lemma 3.1.

Corollary 5.1. *In particular, if*

$$\overline{\lim_{\varepsilon\to0}}\ \fint_{B(z_0,\varepsilon)}\frac{1+|\nu(z)|}{1-|\nu(z)|}\ dxdy\ <\ \infty\qquad\forall\ z_0\in\overline{\mathbb{B}}\ , \tag{37}$$

then the Dirichlet problem

$$\begin{cases} f_{\overline{z}}=\nu(z)\cdot\overline{f_z}, & z\in\mathbb{B},\\ \lim\limits_{z\to\zeta}\mathrm{Re}\,f(z)=\varphi(\zeta), & \forall\ \zeta\in\partial\mathbb{B}\end{cases} \tag{38}$$

in the disk $\mathbb{B}$ has regular solution for each nonconstant continuous function $\varphi:\partial\mathbb{B}\to\mathbb{R}$.

Theorem 5.2. *Let μ, $\nu:\mathbb{B}\to\mathbb{B}$ be measurable functions, $K_{\mu,\nu}\in L^1(\mathbb{B})$, $k_{z_0}(r)$ be the mean value of $K_{\mu,\nu}(z)$ over the circle $|z-z_0|=r$. Suppose that*

$$\int_0^{\delta(z_0)} \frac{dr}{rk_{z_0}(r)} = \infty \qquad \forall\ z_0 \in \overline{\mathbb{B}}\ . \tag{39}$$

Then the Dirichlet problem (7) *in the disk* $\mathbb{B}$ *has regular solution for each nonconstant continuous function* $\varphi : \partial\mathbb{B} \to \mathbb{R}$.

Proof. Theorem 5.2 follows from Lemma 4.1 by special choosing the functional parameter

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := \begin{cases} 1/[tk_{z_0}(t)]\ , & t \in (0, \varepsilon_0)\ , \\ 0\ , & \text{otherwise} \end{cases} \tag{40}$$

where $\varepsilon_0 = \delta(z_0)$.

Corollary 5.2. *In particular, the conclusion of Theorem 5.2 holds if*

$$k_{z_0}(r) = O\left(\log \frac{1}{r}\right) \quad as \quad r \to 0 \qquad \forall\ z_0 \in \overline{\mathbb{B}}\ . \tag{41}$$

In fact, it is clear that the condition (39) implies the whole scale of conditions in terms of *log*, for instance, with using functions of the form $1/(t \log \ldots \log 1/t)$.

In the theory of mappings called quasiconformal in the mean, conditions of the type

$$\int_{\mathbb{B}} \Phi(Q(z))\ dxdy < \infty \tag{42}$$

are standard for various characteristics Q of these mappings.

In this connection, in the paper [26], see also the monograph [14], it was established interconnections between a series of various integral conditions on the function Φ. We give here the corresponding conditions for Φ under which (42) implies (39).

Later on, we use the following notion of the inverse function for monotone functions. Namely, for every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the *inverse function* $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \ge \tau} t\ . \tag{43}$$

As usual, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \ge \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

Remark 5.1. It is evident immediately by the definition that

$$\Phi^{-1}(\Phi(t)) \le t \qquad \forall\ t \in [0, \infty] \tag{44}$$

with the equality in (44) except intervals of constancy of the function Φ.

Further, in (46) and (47), we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, correspondingly, $H(t) = \infty$, for all $t \ge T \in [0, \infty)$. The integral in (47) is understood as the Lebesgue–Stieltjes integral and the integrals (46) and (48)–(51) as the ordinary Lebesgue integrals.

Proposition 5.1. *Let* $\Phi : [0, \infty] \to [0, \infty]$ *be a non-decreasing function and set*

$$H(t) = \log \Phi(t) . \tag{45}$$

Then the equality

$$\int\limits_{\Delta}^{\infty} H'(t) \, \frac{dt}{t} = \infty \tag{46}$$

implies the equality

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty \tag{47}$$

and (47) *is equivalent to*

$$\int\limits_{\Delta}^{\infty} H(t) \, \frac{dt}{t^2} = \infty \tag{48}$$

for some $\Delta > 0$, *and* (48) *is equivalent to every of the equalities:*

$$\int\limits_{0}^{\delta} H\left(\frac{1}{t}\right) \, dt = \infty \tag{49}$$

for some $\delta > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{50}$$

for some $\Delta_* > H(+0)$,

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{51}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (46) *is equivalent to* (47) *and hence* (46)–(51) *are equivalent each to other if* Φ *is in addition absolutely continuous. In particular, all the conditions* (46)–(51) *are equivalent if* Φ *is convex and non-decreasing.*

Finally, we give the connection of the above conditions with the condition of the type (39).

Recall that a function $\psi : [0, \infty] \to [0, \infty]$ is called *convex* if $\psi(\lambda t_1 + (1 - \lambda) t_2) \leq \lambda \psi(t_1) + (1 - \lambda) \psi(t_2)$ for all t_1 and $t_2 \in [0, \infty]$ and $\lambda \in [0, 1]$.

Proposition 5.2. *Let $Q : \mathbb{B} \to [0,\infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{B}} \Phi(Q(z))\, dxdy \ < \ \infty \tag{52}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \ = \ \infty \tag{53}$$

for some $\delta \ > \Phi(0)$. Then

$$\int\limits_{0}^{1} \frac{dr}{rq(r)} \ = \ \infty \tag{54}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Finally, combining Propositions 5.1 and 5.2 we obtain the following conclusion.

Corollary 5.3. *If $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function and Q satisfies the condition* (52)*, then every of the conditions* (46)–(51) *implies* (54)*.*

Theorem 5.2 and Corollary 5.3 imply the following statement

Theorem 5.3. *Let μ and $\nu : \mathbb{B} \to \mathbb{B}$ be measurable functions such that*

$$\int\limits_{\mathbb{B}} \Phi(K_{\mu,\nu}(z))\, dxdy \ < \ \infty \tag{55}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function satisfying at least one of the conditions (46)–(51)*. Then the Dirichlet problem* (7) *in the disk $\mathbb{B}$ has a regular solution for each nonconstant continuous function $\varphi : \partial\mathbb{B} \to \mathbb{R}$.*

Remark 5.1. Note that Theorem 5.50 from the work [26] for the Beltrami equations of the first type (1) shows that the conditions (46)–(51) are not only sufficient but also necessary for the general Beltrami equations (4) with the restriction (55) to have regular solutions to the Dirichlet problem (6) for each nonconstant continuous function $\varphi : \partial\mathbb{B} \to \mathbb{R}$ because by the Stoilow theorem, see e.g. [30], every such a solution F has the representation $F = \varphi \circ f$ where f is a regular homeomorphic solution of (1).

Note also that in the above theorem we may assume that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex and non–decreasing on the whole segment $[0,\infty]$ but only on a segment $[T,\infty]$ for some $T \in (1,\infty)$. Indeed, every function $\Phi : [0,\infty] \to [0,\infty]$ which is convex and non-decreasing on a segment $[T,\infty]$, $T \in (0,\infty)$, can be replaced by a non-decreasing convex function $\Phi_T : [0,\infty] \to [0,\infty]$ in the following way. We set $\Phi_T(t) \equiv 0$ for all $t \in [0,T]$, $\Phi(t) = \varphi(t)$, $t \in [T,T_*]$, and $\Phi_T \equiv \Phi(t)$, $t \in [T_*,\infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0,T)$ and supporting the graph of the function $\tau = \Phi(t)$ at a point $(T_*, \Phi(T_*))$, $T_* \geq T$. For such a function we have by the construction that $\Phi_T(t) \leq \Phi(t)$ for all $t \in [1, \infty]$ and $\Phi_T(t) = \Phi(t)$ for all $t \geq T_*$.

The equation of the form

$$f_{\overline{z}} \;=\; (z)\;\mathrm{Re}\, f_z \tag{56}$$

with $|(z)| < 1$ a.e. is called a *reduced Beltrami equation*, considered e.g. in [2] and [32], though the term is not introduced there. The equation (56) can be written as the equation (4) with

$$\mu(z) \;=\; \nu(z) \;=\; \frac{(z)}{2} \tag{57}$$

and then

$$K_{\mu,\nu}(z) \;=\; K(z) \;:\;=\; \frac{1 + |(z)|}{1 - |(z)|}\;. \tag{58}$$

Thus, we obtain from Theorem 5.3 the following consequence for the reduced Beltrami equation (56).

Theorem 5.4. *Let* $: \mathbb{B} \to \mathbb{B}$ *be a measurable function such that*

$$\int\limits_{\mathbb{B}} \Phi(K(z))\; dxdy \;<\; \infty \tag{59}$$

where $\Phi : [0, \infty] \to [0, \infty]$ *is a non-decreasing convex function satisfying at least one of the conditions* (46)–(51). *Then the Dirichlet problem in the disk* $\mathbb{B}$

$$\begin{cases} f_{\overline{z}} \;=\; (z)\;\mathrm{Re}\, f_z, & z \in \mathbb{B}, \\ \lim\limits_{z \to \zeta} \mathrm{Re}\, f(z) = \varphi(\zeta), & \forall\; \zeta \in \partial\mathbb{B} \end{cases} \tag{60}$$

has a regular solution for each nonconstant continuous function $\varphi : \partial\mathbb{B} \to \mathbb{R}$.

Remark 5.2. Remarks 5.1 are valid for reduced Beltrami equations. Moreover, the above results remain true for the case when in (4)

$$\nu(z) \;=\; \mu(z)\; e^{i\theta(z)} \tag{61}$$

with an arbitrary measurable function $\theta(z) : D \to \mathbb{R}$ and, in particular, for the equations of the form

$$f_{\overline{z}} \;=\; (z)\;\mathrm{Im}\, f_z \tag{62}$$

with a measurable coefficient $: \mathbb{B} \to \mathbb{C}$, $|(z)| < 1$ a.e., see e.g. [3].

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PROBLEM DIRICHLETA DLA OGÓLNYCH ZDEGENEROWANYCH RÓWNAŃ BELTRAMIEGO

S t r e s z c z e n i e

Badamy problem Dirichleta dla ogólnego zdegenerowanego równania Beltramiego $\overline{\partial} f = \mu \partial f + \nu \overline{\partial f}$ w kole jednostkowym. Dowodzimy nowych kryteriów istnienia rozwiązań regularnych.

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In memory of
Professor Promarz M. Tamrazov

Ralitza K. Kovacheva

RATIONAL CHEBYSHEV APPROXIMANTS – THE REAL CASE, SURVEY

Summary

In the present paper, a survey about convergence results and distribution of zeros, free poles and alternation points of rational functions of best uniform rational approximants – the real case is provided.

Keywords and phrases: Chebyshev approximant, distribution of zeros and free poles, distribution of alteration points

1. Introduction

Let Δ be a real segment, say $\Delta := [-1, 1]$. Given a positive integer $n (n \in \mathbb{N})$, we set Π_n for the class of all polynomials of degree $\leq n$. For each pair $(n, m), n, m \in \mathbb{N} \bigcup \{0\}$ we introduce $R_{n,m} := \{R, R = p/q, p \in \Pi_n, q \in \Pi_m, q \neq 0\}$. Let now f be a function continuous and real-valued on Δ $(f \in C_r(\Delta))$. Throughout the paper, we shall be dealing with nonrational functions f.

We further introduce the value of $\rho_{n,m}$ as the minimal deviation of f from the class $R_{n,m}$ on the segment Δ, that is:

$$\rho_{n,m} := \inf_{R \in R_{n,m}} ||f - R||_\Delta,$$

where the norm is taken in the Chebyshev (max) metric on Δ. Let $r_{n,m} \in R_{n,m}$ be a real-valued function on Δ such that

$$||f - r_{n,m}||_\Delta = \rho_{n,m}$$

The function $r_{n,m}$ is called *rational function of best Chebyshev approximation of* f *in the class* $R_{n,m}$*;* as known [1], it always exists and is uniquely determined by the alternation theorem of Chebyshev. By the classical Weierstrass' approximation theorem [9], for instance $\rho_{n,m} \to 0$ as $n+m \to \infty$ (recall that $f \in C_r(\Delta)$).

The theory of best Chebyshev approximants plays an important role in the modern mathematical analysis, because of its strong relation to the logarithmic potential theory. In the present paper, we concentrate ourselves at the subjects of *convergence of Chebyshev rational approximants* and of the *distribution of* α*-points, free poles and points of alternations.* The convergence consists of two components: the *inverse problems* which refer to characterizing functions using an information about the behavior of rational Chebyshev approximants and *direct problems* what means to get a look into the asymptotic behavior of sequences of best Chebyshev approximants going out of properties of the approximated function f itself.

Before stating our considerations, we introduce some notations. Let

$$\phi(z) := z + \sqrt{z^2 - 1} \quad \text{with} \quad \phi(\infty) = \infty.$$

For each $r, r > 1$, let

$$E_r := \{z, |\phi(z)| < r\} \text{ and } \Gamma_r := \partial E_r$$

be the Joukovsky ellipse with parameter r.

Given a pair (n, m), let

$$r_{n,m} := p_{n,m}/q_{n,m},$$

where both polynomials $p_{n,m}$ and $q_{n,m}$ have no common divisors. The zeros $\alpha_{n,m,k}$, $k \leq l(n,m) \leq m$ of $q_{n,m}$ are called *the free* poles of $r_{n,m}$.

The next theorem is the basis for rational Chebyshev approximation (the real case).

Alternation Theorem by Chebyshev, [1]. *Given* $f \in C_r(\Delta)$ *and* (n, m) *a fixed pair, let* $r_{n,m}$ *be the rational function of best Chebyhev approximation of* f *on* Δ *in the class* $\mathcal{R}_{n,m}$*. Then there exist at least* $n + m + 2 - d_{n,m}$ *points*

$$-1 \leq \zeta_0^{(n,m)} < \cdots < \zeta_k^{(n,m)} < \cdots \zeta_{n+m-d_{(n,m)}}^{(n,m)} <$$

$$< \zeta_{n+m-d_{(n,m)}+1}^{(n,m)} \leq 1$$

such that

$$(f - r_{n,m}(\zeta_k^{(n,m)})) =$$

$$(-1)^k \rho_{n,m} (-1)^\delta,\ k = 0, \cdots, n+m-d_{n,m}+1,\ \delta = \pm 1,$$

where $d_{n,m} = \min\left(n - \deg p_{n,m}, m - \deg q_{n,m}\right)$.

The extreme points $\{\zeta_k^{(n,m)}\}$ are called *alternation points* of $f - r_{n,m}$.

As well known, the sequence of those pairs, for which $d_{n,m} = 0$ is infinite iff f is not a rational function. Hence, one may assume without losing the generality (should a situation arise) that $d_{n,m} = 0$ for every (n, m).

To each function $f \in C_r(\delta)$ we assign the s.c. *Walsh's table* $:= W(f)$, that is:

$$W(f) := \begin{pmatrix} r_{0,0}, & r_{1,0}, & \cdots, & r_{n,0}, & \cdots \\ r_{0,1}, & r_{1,1}, & \cdots, & r_{n,1}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots, & \cdots \\ r_{0,m}, & r_{1,m}, & \cdots, & r_{n,m}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots, & \cdots \end{pmatrix}$$

The sequence $\{r_{n,m}\}$, m-fixed, $n \to \infty$ is called *the row* in Walsh's table, the sequence $\{r_{n,n}\}$, $n \to \infty$ – the diagonal, and $\{r_{n,m_n}\}, m_n = o(n)$, $\{r_{n,m_n}\}, m_n/n \to c$, $c > 0$ – *a closed to the row* and a *ray-sequence,* respectively.

2. Direct results

We start with some notations and definitions.

For $B \subset \mathbb{C}$, we denote by $C(B)$ the class of *continuous functions* on B and $\mathcal{A}(B)$ (resp. $\mathcal{M}(B)$) represents the class of functions f that are holomorphic – analytic and single valued (resp. meromorphic) in B. Moreover, we will denote by $\mathcal{M}_m(B)$ the subset of functions f of $\mathcal{M}(B)$ with no more than m poles in B, each pole counted with its multiplicity. To each function $f \in C(\Delta)$ we associate the *radius of holomorphy* $\rho_0(f)$, that is:

$$\rho_0(f) = \sup\{\rho \geq 1, f \in \mathcal{A}(E_\rho)\}$$

and, analogously, the radius of m-meromorphy $\rho_m(f)$ and the radius of meromorphy ρ_f. If $f \in C_r(\Delta)$ has a nonpolar singularity on Δ, then we set $\rho_0(f) = 1$.

By the classical theorem of Weierstrass, $r_{n,m}(x) \to f(x)$, $x \in \Delta$ as $n + m \to \infty$. We pose the question about the behavior of sequences of rational Chebyshev approximants *outside* the interval Δ.

Theorem 1 [21]. *Assume that the function $f \in C_r(\Delta) \bigcap \mathcal{M}_m(E_r)$, $r > 1$, has exactly m poles in the ellipse E_r. Then each pole of f attracts as many poles of the sequence $\{r_{n,m}\}$, $n \to \infty$, m-fixed, as its multiplicity; in the remaining domain the sequence $\{r_{n,m}\}$ converges to f uniformly on compact subsets. More precisely, the sequence $\{r_{n,m}\}$, $n \to \infty$ converges in m_1-measure to f inside E_r.*

This result of Walsh generalizes Bernsteins' result about the convergence of best Chebyshev polynomial approximants to f inside the ellipse of holomorphy $E_{\rho_o(f)}$. Furthermore, Theorem 1 is a partial case of Gonchar's theorem about the rate of best uniform approximation with rational function with fixed number if the free poles [15].

Things are not so good when the degrees of the denominators increase. Before presenting Theorem 2 we remind the reader of the term of *an almost uniform m_1-convergence* inside some domain B [10]. A sequence of functions $\{\varphi_n\}$, meromorphic in B, is said to converge to a function φ *m_1-almost uniformly inside* B if for any compact set $K \subset B$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and the sequence $\{\varphi_n\}$ converges uniformly to φ on K_ε.

Theorem 2 [5]. *Assume that $f \in C_r(\Delta) \bigcap \mathcal{A}(\Delta)$, and assume that $\rho_f < \infty$. Let $\{m_n\}$ be a sequence of positive integers such that*

$$m_n \leq n,\, m_n \leq m_{n+1} \leq m_n + 1$$

and

$$m_n = o(n/\log n),\, n \to \infty.$$

Then there is a subsequence $\Lambda \subset \mathbf{N}$ such that the sequence $\{r_{n,m_n}\}$ converges m_1-almost uniformly to f inside E_{ρ_f}. More precisely, the sequence Λ is that sequence of positive integers for which

$$\frac{\rho_{n,m_m} - \rho_{n+1,m_{n+1}}}{\rho_{n,m_m} + \rho_{n+1,m_{n+1}}} \geq \frac{1}{n^2}.$$

We introduce the next result established by H. Stahl about the diagonal in the Walsh's table:

Theorem 3 [19]. *Suppose that the function $f \in C_r(\Delta) \bigcap \mathcal{A}(\Delta)$ admits an analytic continuation everywhere in the extended complex plane $\mathbb{C}$ except for a finite number of multivalued singularities $a_i, i = 1, \dots j$. Then there is a regular compact set F, $F \bigcap \Delta = \emptyset$, $F \supset \{a_i\}_{i=1,\dots j}$ such that for every compact set $\mathbb{C} \setminus (F \bigcup \Delta)$*

$$||f(z) - r_{n,n}(z)||_K^{1/n} \longrightarrow^{\text{cap}} e^{-2G_F^\mu(z)} < 1,$$

where $\longrightarrow^{cap}$ means a convergence in capacity on K, μ – the equilibrium measure of the interval Δ,

$$d\mu = \frac{1}{2\pi} \frac{dx}{\sqrt{1-x^2}}$$

and G_F^μ – the Green's potential of Δ with respect to F.

The compact set F, called *the stationary compact set*, is unique and consists of a finite number of disjoint curves nonintersecting the interval Δ.

For the most general case, we have, so far

Theorem 4 [13]. *Let $f \in C_r(\Delta)$ be analytic in the ellipse $E_r, r > 1$. Assume that $r_{n,n} \in \mathcal{A}(E_r)$ for all n from some n_0. Then the sequence $\{r_{n,n}\}$ converges to f as $n \to \infty$ inside the ellipse E_ρ for every ρ with*

$$\frac{(r\rho - 1)^2}{r(r-\rho)^2} < 1.$$

3. Inverse results

As a first result in this direction, we mention

Theorem 5 [18]. *Given $f \in C_r(\Delta)$, assume that there is a polynomial q of degree exactly m, $q(z) = \prod_{i=1}^{m}(z - a_i)$ with no zeros on Δ such that*

$$\limsup_{n\to\infty} ||q - q_{n,m}||^{1/n} = \sigma < 1.$$

Then f admits a meromorphic continuation with exactly m poles into the ellipse E_R with

$$R \geq \max_{1\leq i\leq m} \frac{|\phi(a_i)|}{\sigma}$$

and all point $a_i, i = 1, \ldots, m$ are poles of f.

Given now an $n \in \mathbb{N}$, we introduce the set $\alpha_n := \{\alpha_{n,m,k}\}_{k=1}^{l_n}$. Let L and l be the sets of accumulation points of $\{\alpha_n\}$ and of limit points, respectively, as $n \to \infty$. The next result is due to K. Lungu.

Theorem 6 [16]. *Given a function $f \in C_r(\Delta)$, assume that $L = l$ with L being a finite set, nonintersecting the interval Δ. Then f admits a holomorphic continuation from Δ into $\mathbb{C} \setminus L$.*

Given now a point $a \in \mathbb{C}$, an integer n and a positive number ε, we introduce the value of $\tau_n(a, \varepsilon)$ as the number of all free poles of r_n which are lying in the circle $K_a(\varepsilon) := \{z, |z - a| < \varepsilon\}$. Finally, set

$$\tau(a, \varepsilon) := \lim_{n\to\infty} \inf \frac{\tau_n(a, \varepsilon)}{n} \quad \text{and} \quad \tau(a) := \lim_{\varepsilon\to 0} \tau(a, \varepsilon).$$

Obviously, if for some point a we have $\tau(a) > 0$, then $a \in l$.

The following result was announced by Lungu:

Theorem 7 [16]. *Let $f \in C_r(\Delta)$. Assume that $\Delta \cap L = \emptyset$ and that the set L does not separate the plane. If there is a point $a \in \mathbb{C}$ such that*

$$\tau(a) > 0,$$

then the function f admits a holomorphic continuation from Δ into $\overline{\mathbb{C}}\setminus L$.

Sufficient conditions under the conditions of Theorems 6 and 7 for the function f to have a pole at some point $a \in L$ were given in [14].

Information about the holomorphic continuation of functions in terms of the associated rational Chebyshev approximants could be provided by the behavior of the zeros. Before continuing, we introduce the term of $N_\alpha(g, A)$. Given a set A, function g meromorphic in A and $\alpha \in \overline{\mathbb{C}}$, we denote by $N_\alpha(g, A)$ the number of the α-points of g in A. Under this definition, $N_0(g, A)$ stands for the number of the zeros of g in A, whereas $N_\infty(g, A)$ means the number of the poles.

Theorem 8 [12]. *Let $f \in C_r(\Delta)$. Suppose that all $r_{n,n} \in \mathcal{M}_m(U)$ for some domain $U \supset \Delta$. Suppose further that for any compact subset K of U*

$$N_0(r_{n,n}, K) = o(n),\ n \to \infty.$$

Then $f \in \mathcal{M}_m(U)$ and the sequence $\{r_{n,n}\}$ converges to f, as $n \to \infty$ m_1-almost uniformly inside U.

4. Distribution of alternation point, of α-points and of free poles

Recall the classical theorem of Kadec [11].

Theorem 9 [11]. *Given $f \in C_r(\Delta)$ and n a fixed positive integer. Let P_n be the polynomial of best Chebyshev approximation of f on Δ. Denote by $\{x_k\}_{k=0}^{n+1}$ the set of the alternation points of $f - P_n$. Then there is a sequence $\Lambda \subset \mathbf{N}$ such that for every positive ε*

$$\lim_{n \in \Lambda} (\max_{0 \le k \le n+1} |x_k - \cos \frac{\pi k}{n+1}|) \le \frac{1}{n^{1/2-\varepsilon}}.$$

Denote by ν_n the unit (counting) measures associated with the polynomials of best Chebyshev approximation. As a consequence, we get:

Corollary 1 . *Under the conditions of Kadec's theorem, there is a sequence Λ such that*

$$\nu_n \Longrightarrow \mu \; as \, n \in \Lambda,$$

where μ is the equilibrium measure on Δ and $\Longrightarrow$ means convergence of measures in the weak sense.

We remark that the sequence Δ is that one, for which

$$\frac{\rho_{n,0} - \rho_{n+1,0}}{\rho_{n,0} + \rho_{n+1,0}} \ge \frac{1}{n^2},\ n \in \Lambda$$

(comp. footnote 2), p. 4).

The next result was proved by H. P. Blatt, R. Grothmann and R. K. Kovacheva [3].

Theorem 10 . *Let $f \in C_r(\Delta)$ and*

$$m_n \le n, m_n + 1 \le m_{n+1} \le m_n + 1, n = 1, 2, \cdots .$$

Then for the sequence sequence Λ

$$\Lambda := \{n \in \mathbf{N}, \frac{\rho_{n,m_n} - \rho_{n+1,m_{n+1}}}{\rho_{n,m_n} + \rho_{n+1,m_{n+1}}} \ge \frac{1}{n^2}\}$$

there holds

$$\nu_{n,m_n} - \alpha_{n,m_n} \tau^{(b)}_{n,m_n} - (1 - \alpha_{n,m_n})\mu \Longrightarrow 0, \quad as \quad n \in \Lambda,$$

where ν_{n,m_n} is the unit measures associated with the alternation points of $f - r_{n,m_n}$, $\tau^{(b)}_{n,m_n}$ – the ballayage onto the segment Δ of the unit measure, associated with the polynomial $q_{n,m_n}q_{n+1,m_{n+1}}$ and

$$\alpha_{n,m_n} = \frac{m_n + m_{n+1}}{n + m_n + 2}.$$

Theorem 9 generalizes the result of Kadec for $m_n = o, n = 1, 2, \ldots$, as well as the results in [6] and [7] for closed to row sequences, ray-sequences and diagonal sequences in the table of Walsh, respectively.

Given an $f \in C_r(\Delta)$, denote by $\mathcal{E}_f$ the ellipse of meromorphy of f, that is the maximal Joukovsky ellipse with foci at ± 1 into which f admits a continuation as a meromorphic function. Given a set A and point $\alpha \in \overline{\mathbb{C}}$, we denote by $N_\alpha(r_{n,m_n}, A)$ the number of the α-points of r_{n,m_n} in A.

The next theorem [12] and [4] is an *Analogue of Sohotzky's Theorem* about the behavior of α-points of a function around an essential singularity:

Theorem 11 . *Assume that $f \in C_r(\Delta)$ is holomorphic (analytic and single valued) on Δ. Assume, further, that $m_n = o(n/\log n)$ as $n \to \infty$. Then the sequence r_{n,m_n} converges in m_1-measure to f as $n \in \Lambda$, uniformly on compact subsets of the ellipse $\mathcal{E}_f$. Let $z_0 \in \partial\mathcal{E}_f$ be a point through f does not admit a holomorphic continuation, and let U be a neighborhood of z_0. Then for every number $\alpha \in \overline{\mathbb{C}}$ with at most one exception there holds*

$$\limsup N_\alpha(r_{n,m_n}, U) = \infty.$$

Theorem 12 introduced below could be considered as an analogue of Jentzsch-Szegö theorem about the asymptotic distribution of zeros of approximation polynomials.

Theorem 12 [5]. *Under the same conditions on f and $\{m_n\}$ as in Theorem 2, assume that f has at least one singularity on $\partial\mathcal{E}_f$ of multivalued character. Then the zeros of the sequence r_{n,m_n} distribute asymptotically, as $n \in \Lambda$, like the equilibrium measure of the boundary of $\mathcal{E}_f$.*

In the spirit of Theorem 12, we consider the case, when f is not holomorphic on Δ. In this case, we have

Theorem 13 Andrievsky-Blatt-Kovacheva, [2]. *Let $a \in \Delta$ be a point of non-holomorphy of f. Then for every neighborhood $U, U \bigcap \Delta \subset \Delta$ and any $a \in \mathbb{C}$ there holds:*

$$\begin{array}{ll} \textit{either} & \limsup N_\infty(r_{n,m_n}, U) = \infty, \\ \textit{or} & \limsup \frac{N_a(r_{n,m_n}, U)}{n} > 0. \end{array}$$

This result generalizes results by Blatt, Iserles, Saff and Stahl about distribuution of zeros and poles of Chebyshev approximants for the functions $|x|$ and $x^\alpha, 1 > \alpha > 0$.

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WYMIERNE APROKSYMANTY CZEBYSZEWA – – PRZYPADEK RZECZYWISTY. PRZEGLĄD

S t r e s z c z e n i e

W niniejszej pracy dokonujemy przeglądu wyników dotyczących rozmieszczenia zer, swobodnych biegunów i punktów alteracji funkcji wymiernych z najlepszej jednorodnej aproksymanty wymiernej w przypadku rzeczywistym.

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In memory of
Professor Promarz M. Tamrazov

Sergiy A. Plaksa and Vitalii S. Shpakivskyi

A DESCRIPTION OF SPATIAL POTENTIAL FIELDS BY MEANS OF MONOGENIC FUNCTIONS IN INFINITE-DIMENSIONAL SPACES WITH A COMMUTATIVE MULTIPLICATION

Summary

We consider monogenic functions taking values in topological vector spaces being expansions of certain infinite-dimensional commutative Banach algebras associated with the three-dimensional Laplace equation. We establish that every harmonic function is a component of the mentioned monogenic functions

Keywords and phrases: Laplace equation, harmonic vector, harmonic function, harmonic commutative Banach algebra, topological vector space, monogenic function, Cauchy-Riemann conditions

1. Introduction

An effectiveness of the analytic function methods in the complex plane for researching plane potential fields inspires mathematicians to develop analogous methods for spatial fields.

An algebraic-analytic approach to equations of mathematical physics is developed at the Department of Complex Analysis and Potential Theory of the Institute of Mathematics of the National Academy of Sciences of Ukraine. Being the first head of this Department, Professor P. M. Tamrazov concerned very closely to development of the mentioned approach that were essentially developed thanking his support.

This approach means a finding of commutative Banach algebra such that functions differentiable in the sense of Gâteaux with values in this algebra have components satisfying the given equation with partial derivatives. Such algebras are constructed for the biharmonic equation and the three-dimensional Laplace equation and elliptic equations degenerating on an axis that describe axial-symmetric potential fields (see [1–5]).

Defining a *potential solenoid* field in a simply connected domain Q of the three-dimensional real space $\mathbb{R}^3$, the vector-function $\mathbf{V}$ satisfies the system of equations

$$\operatorname{div}\mathbf{V} = 0\,, \qquad \operatorname{rot}\mathbf{V} = 0\,. \tag{1}$$

Then there exists a scalar potential function $u(x, y, z)$ such that $\mathbf{V} = \operatorname{grad} u$ and u satisfies the three-dimensional Laplace equation

$$\Delta_3 u(x, y, z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x, y, z) = 0\,. \tag{2}$$

Doubly continuously differentiable functions satisfying Eq. (2) are called *harmonic functions*, and solutions of the system (1) are called *harmonic vectors.*

Let $\mathbb{A}$ be a n-dimensional commutative associative Banach algebra over either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, $3 \le n \le \infty$. Let $\{e_1, e_2, e_3\}$ be a part of the basis of $\mathbb{A}$ and $E_3 := \{\zeta = xe_1 + ye_2 + ze_3 \,:\, x, y, z \in \mathbb{R}\}$ be the linear span generated by the vectors e_1, e_2, e_3. In what follows, $\zeta := xe_1 + ye_2 + ze_3$ and $x, y, z \in \mathbb{R}$.

We say that a continuous function $\Phi : \Omega_\zeta \to \mathbb{A}$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i.e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3. \tag{3}$$

We use the notion of monogenic function in the sense of existence of derived numbers for this function in the domain D_ζ (cf. [6, 7]).

In the scientific literature the denomination of monogenic function is used else for functions satisfying certain conditions similar to the classical Cauchy – Riemann conditions (cf. [8, 9]). Such functions are also called regular functions (cf. [10]) or hyperholomorphic functions (cf. [11, 12]).

If the basic elements e_1, e_2, e_3 satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{4}$$

then every doubly differentiable in the sense of Gateaux function $\Phi : \Omega_\zeta \to \mathbb{A}$ satisfies the following equality in the domain Ω_ζ:

$$\Delta_3 \Phi(\zeta) \equiv \Phi''(\zeta)\ (e_1^2 + e_2^2 + e_3^2) = 0\,. \tag{5}$$

We say that an algebra $\mathbb{A}$ is *harmonic* (cf. [3, 4, 13]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors $\{e_1, e_2, e_3\}$ satisfying the equality (4) provided that $e_k^2 \ne 0$ for $k = 1, 2, 3$. We say also that such a triad $\{e_1, e_2, e_3\}$ is *harmonic.*

P. W. Ketchum [13] considered the C. Segre algebra of quaternions [14] as an example of harmonic algebra. I. P. Mel'nichenko solved completely the problem on finding three-dimensional harmonic algebras with unit (see [1, 3, 4]). More exactly, he proved that there does not exist such an algebra over the field $\mathbb{R}$, he found all three-dimensional harmonic algebras over the field $\mathbb{C}$ and constructed all harmonic bases in these algebras.

Yet, it is the fact that it is impossible to obtain all solutions of Eq. (2) in the form of components of monogenic functions taking values in finite-dimensional commutative algebras (see, e.g., [4, p. 43]).

Meanwhile, it is proved in [4] that potential functions of axial-symmetric fields are expressed in the form of components of monogenic functions taking values in an infinite-dimensional commutative Banach algebra if certain natural requirements are fulfilled. Spherical functions are the first components of expansions of corresponding monogenic functions with respect to the basis of infinite-dimensional commutative Banach algebra $\mathbb{F}$ considered in the papers [4, 5]. Note that the algebra $\mathbb{F}$ is contained in a vector space considered by P. W. Ketchum [15]. This vector space is not an algebra, though. P. W. Ketchum proved that a set of components of functions taking values in the mentioned space includes all analytic solutions of Eq. (2). M. N. Roşculeţ [16] considered an other infinite-dimensional vector space and functions generating solutions of Eq. (2).

Below, we consider a topological vector space containing the algebra $\mathbb{F}$ and prove that all harmonic functions are components of monogenic functions taking values in this space. We prove also a similar result for monogenic functions taking values in a topological vector space whose support coincides with the vector space of the paper [16].

2. A topological vector space $\widetilde{\mathbb{F}}$ containing the algebra $\mathbb{F}$

Consider an infinite-dimensional commutative associative Banach algebra

$$\mathbb{F} := \left\{ g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{R}, \sum_{k=1}^{\infty} |c_k| < \infty \right\}$$

over the field $\mathbb{R}$ with the norm $\|g\|_{\mathbb{F}} := \sum\limits_{k=1}^{\infty} |c_k|$ and the basis $\{e_k\}_{k=1}^{\infty}$, where the multiplication table for elements of basis is of the following form:

$$e_n e_1 = e_n, \qquad e_{2n+1} e_{2n} = \frac{1}{2}\, e_{4n} \quad \forall\, n \geq 1\,,$$

$$e_{2n+1} e_{2m} = \frac{1}{2}\Big(e_{2n+2m} - (-1)^m e_{2n-2m}\Big) \quad \forall\, n > m \geq 1\,,$$

$$e_{2n+1} e_{2m} = \frac{1}{2}\Big(e_{2n+2m} + (-1)^n e_{2m-2n}\Big) \quad \forall\, m > n \geq 1\,,$$

$$e_{2n+1} e_{2m+1} = \frac{1}{2}\Big(e_{2n+2m+1} + (-1)^m e_{2n-2m+1}\Big) \quad \forall\, n \geq m \geq 1\,,$$

$$e_{2n} e_{2m} = \frac{1}{2}\Big(-e_{2n+2m+1} + (-1)^m e_{2n-2m+1}\Big) \quad \forall\, n \geq m \geq 1\,.$$

It is evident that here e_1, e_2, e_3 form a harmonic triad of vectors.

Note that the algebra $\mathbb{F}$ is isomorphic to the algebra $\mathbf{F}$ of absolutely convergent trigonometric Fourier series

$$g(\tau) = a_0 + \sum_{k=1}^{\infty}\Big(a_k\, i^k \cos k\tau + b_k\, i^k \sin k\tau\Big)$$

with real coefficients a_0, a_k, b_k and the norm $\|g\|_{\mathbf{F}} := |a_0| + \sum\limits_{k=1}^{\infty}\Big(|a_k| + |b_k|\Big)$. In this case, we have the isomorphism $e_{2k-1} \leftrightarrow i^{k-1}\cos(k-1)\tau$, $e_{2k} \leftrightarrow i^k \sin k\tau$ between basic elements.

Now, let us to insert the algebra $\mathbb{F}$ in the topological vector space

$$\widetilde{\mathbb{F}} := \left\{ g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{R} \right\}$$

with the topology of coordinate-wise convergence.

Essentially, P. W. Ketchum [15] considered the space $\widetilde{\mathbb{F}}$ though he did not use the notion of topological vector space as well as the differentiability in the sense of Gateaux.

Note that $\widetilde{\mathbb{F}}$ is not an algebra because the product of elements $g_1, g_2 \in \widetilde{\mathbb{F}}$ is defined not always. But for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{F}}$ and $\zeta = xe_1 + ye_2 + ze_3$, one can define the product

$$g\zeta \equiv \zeta g := x\sum_{k=1}^{\infty} c_k e_k + y\Bigg(-\frac{c_2}{2}\, e_1 + \left(c_1 - \frac{c_5}{2}\right) e_2 - \frac{c_4}{2}\, e_3 +$$

$$+\frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-1} - c_{2k+3})\, e_{2k} - \frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-2} + c_{2k+2})\, e_{2k+1}\Bigg) +$$

$$+z\Bigg(-\frac{c_3}{2}\, e_1 - \frac{c_4}{2}\, e_2 + \left(c_1 - \frac{c_5}{2}\right) e_3 + \frac{1}{2}\sum_{k=4}^{\infty}(c_{k-2} - c_{k+2})\, e_k\Bigg).$$

Let Ω be a domain in $\mathbb{R}^3$ and $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\}$. Consider a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{F}}$ of the form

$$\Phi(\zeta) = \sum_{k=1}^{\infty} U_k(x, y, z)\; e_k\,, \tag{6}$$

where the functions $U_k : \Omega \to \mathbb{R}$ are differentiable in Ω. Then Φ is a continuous function in Ω_ζ and, therefore, Φ is a monogenic function in Ω_ζ if $\Phi'(\zeta) \in \widetilde{\mathbb{F}}$ in the equality (3).

In the following theorem we establish necessary and sufficient conditions for a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{F}}$ to be monogenic in a domain Ω_ζ.

Theorem 2.1. *Let a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{F}}$ be of the form* (6) *and the functions $U_k : \Omega \to \mathbb{R}$ be differentiable in Ω. In order that the function Φ be monogenic in* *the domain Ω_ζ, it is necessary and sufficient that the following Cauchy-Riemann conditions be satisfied in Ω_ζ:*

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x}\, e_2, \qquad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\, e_3. \tag{7}$$

The proof of Theorem 2.1 is similar to the proof of Theorem 1.16 [5] where the necessary and sufficient conditions for a function $\Phi : \Omega_\zeta \to \mathbb{F}$ to be monogenic include certain additional relations conditioned by the norm of absolute convergence in the algebra $\mathbb{F}$.

It is proved in [5] that the system (7) can be rewritten in the following equivalent form:

$$\begin{aligned}
&\frac{\partial U_1}{\partial x} - \frac{1}{2}\frac{\partial U_2}{\partial y} - \frac{1}{2}\frac{\partial U_3}{\partial z} = 0\,,\\
&\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} = 0\,,\\
&\frac{\partial U_1}{\partial y} + \frac{1}{2}\frac{\partial U_2}{\partial x} = 0\,,\\
&\frac{\partial U_1}{\partial z} + \frac{1}{2}\frac{\partial U_3}{\partial x} = 0\,,\\
&\frac{\partial U_{2k}}{\partial x} = -\frac{\partial U_{2k-2}}{\partial z} - \frac{\partial U_{2k-1}}{\partial y}\\
&\frac{\partial U_{2k+1}}{\partial x} = \frac{\partial U_{2k-2}}{\partial y} - \frac{\partial U_{2k-1}}{\partial z}\\
&\frac{\partial U_{2k}}{\partial z} - \frac{\partial U_{2k+1}}{\partial y} = \frac{\partial U_{2k-2}}{\partial x}\\
&\frac{\partial U_{2k}}{\partial y} + \frac{\partial U_{2k+1}}{\partial z} = \frac{\partial U_{2k-1}}{\partial x} \qquad k = 2, 3, \ldots\,.
\end{aligned} \tag{8}$$

Relations between solutions of the system (8) and spatial potential fields is described in the paper [5]. In particular, the following theorem is a direct consequence of Theorem 1.19 [5].

Theorem 2.2. *Every monogenic function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{F}}$ generates a harmonic vector $\mathbf{V} := (U_1, -\frac{1}{2}\, U_2, -\frac{1}{2}\, U_3)$ in the domain Ω.*

Note that the same relation between harmonic vectors and monogenic functions taking values in a three-dimensional semisimple harmonic algebra was discovered by I. P. Mel'nichenko [1,4].

The following theorem is a direct consequence of Theorem 1.20 [5].

Theorem 2.3. *For every function $U_1 : \Omega \to \mathbb{R}$ harmonic in a simply connected domain $\Omega \subset \mathbb{R}^3$ there exist a monogenic function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{F}}$ such that U_1 is the first component of the expansion* (6).

In particular, every spherical function is the first component of the expansion (6) of monogenic function taking values in the algebra $\mathbb{F}$, videlicet, $\Phi(\zeta) = a\zeta^n$, where $a \in \mathbb{F}$ (see, e.g., [5]).

3. A topological vector space containing another infinite-dimensional harmonic algebra

Consider an infinite-dimensional commutative associative Banach algebra

$$\mathbb{G} := \left\{ g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{R}, \sum_{k=1}^{\infty} |c_k| < \infty \right\}$$

over the field $\mathbb{R}$ with the norm $\|g\|_{\mathbb{G}} := \sum\limits_{k=1}^{\infty} |c_k|$ and the basis $\{e_k\}_{k=1}^{\infty}$, where the multiplication table for elements of basis is of the following form:

$$e_n e_1 = e_n, \quad e_{2n+1} e_m = e_{2n+m}, \quad e_{2n} e_{2m} = -e_{2n+2m-3} - e_{2n+2m+1} \tag{9}$$

for all natural numbers n and m.

It is evident that here e_1, e_2, e_3 form a harmonic triad of vectors.

Setting the following correspondence between basic elements e_k and trigonometric functions: $e_{2n-1} \leftrightarrow i^{n-1} \cos^{n-1} \tau$, $e_{2n} \leftrightarrow i^n \sin\tau \cos^{n-1} \tau$, one can obtain a model of the algebra $\mathbb{G}$.

Let Ω and Ω_ζ denote the same domains as above. Consider a function $\Phi : \Omega_\zeta \to \mathbb{G}$ of the form (6), where the functions $U_k : \Omega \to \mathbb{R}$ are differentiable in Ω. In the following theorem we establish necessary and sufficient conditions for such a function Φ to be monogenic in a domain Ω_ζ.

Theorem 3.1. *Let a function $\Phi : \Omega_\zeta \to \mathbb{G}$ be continuous in a domain Ω_ζ and the functions $U_k : \Omega \to \mathbb{R}$ in the expansion* (6) *be differentiable in Ω. In order that the function Φ be monogenic in Ω_ζ, it is necessary and sufficient that the conditions* (7) *be satisfied and the following relations be fulfilled in Ω:*

$$\sum_{k=1}^{\infty} \left| \frac{\partial U_k(x,y,z)}{\partial x} \right| < \infty, \tag{10}$$

$$\lim_{\varepsilon \to 0+0} \sum_{k=1}^{\infty} \left| U_k(x + \varepsilon h_1, y + \varepsilon h_2, z + \varepsilon h_3) - U_k(x,y,z) - \frac{\partial U_k(x,y,z)}{\partial x} \varepsilon h_1 - \right.$$

$$\left. - \frac{\partial U_k(x,y,z)}{\partial y} \varepsilon h_2 - \frac{\partial U_k(x,y,z)}{\partial z} \varepsilon h_3 \right| \varepsilon^{-1} = 0 \quad \forall h_1, h_2, h_3 \in \mathbb{R}. \tag{11}$$

The proof of Theorem 3.1 is similar to the proof of Theorem 1.16 [5]. The relations (10) and (11) are conditioned by the norm of absolute convergence in the algebra $\mathbb{G}$.

Probably, it is impossible to obtain all harmonic functions in the form of components of monogenic functions taking values in the algebra $\mathbb{G}$. Therefore, let us to insert the algebra $\mathbb{G}$ in the topological vector space

$$\widetilde{\mathbb{G}} := \left\{ g = \sum_{k=-\infty}^{\infty} c_k e_k : c_k \in \mathbb{R} \right\}$$

with the topology of coordinate-wise convergence and the basis $\{e_k\}_{k=-\infty}^{\infty}$. We set that the elements of basis are multiplied by rules (9) for all integer numbers n and m.

Essentially, M. N. Roşculeţ [16] considered the space $\widetilde{\mathbb{G}}$ though he did not use the notion of topological vector space as well as the differentiability in the sense of Gateaux. It is proved in [16] that every spherical function is a component of the expansion with respect of the basis of function $\lambda\zeta^n$, where $\lambda \in \widetilde{\mathbb{G}}$.

Note that $\widetilde{\mathbb{G}}$ is not an algebra because the product of elements $g_1, g_2 \in \widetilde{\mathbb{G}}$ is defined not always. But, at least, for each $g = \sum\limits_{k=-\infty}^{\infty} c_k e_k \in \widetilde{\mathbb{G}}$ and $\zeta = xe_1 + ye_2 + ze_3$, one can define the product

$$g\zeta \equiv \zeta g := x \sum_{k=-\infty}^{\infty} c_k e_k +$$

$$+ y \left(\sum_{k=-\infty}^{\infty} c_{2k-1}\, e_{2k} - \sum_{k=-\infty}^{\infty} (c_{2k-2} + c_{2k+2})\, e_{2k+1} \right) + z \sum_{k=-\infty}^{\infty} c_{k-2}\, e_k \,.$$

Now, consider a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{G}}$ of the form

$$\Phi(\zeta) = \sum_{k=-\infty}^{\infty} U_k(x, y, z)\; e_k \,, \tag{12}$$

where the functions $U_k : \Omega \to \mathbb{R}$ are differentiable in Ω. Then Φ is a continuous function in Ω_ζ and, therefore, Φ is a monogenic function in Ω_ζ if $\Phi'(\zeta) \in \widetilde{\mathbb{G}}$ in the equality (3).

In the following theorem we establish necessary and sufficient conditions for a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{G}}$ to be monogenic in a domain Ω_ζ.

Theorem 3.2. *Let a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{G}}$ be of the form* (12) *and the functions $U_k : \Omega \to \mathbb{R}$ be differentiable in Ω. In order that the function Φ be monogenic in the domain Ω_ζ, it is necessary and sufficient that the conditions* (7) *be satisfied in Ω_ζ.*

The proof of Theorem 3.2 is similar to the proof of Theorem 1.16 [5] but relations of the form (10) and (11) are needless.

Let us to rewrite the conditions (7) in expanded form:

$$\begin{aligned}
&\frac{\partial U_{2m+2}}{\partial y} = \frac{\partial U_{2m+1}}{\partial x}\,, \\
&\frac{\partial U_{2m+1}}{\partial y} = -\frac{\partial U_{2m-2}}{\partial x} - \frac{\partial U_{2m+2}}{\partial x}\,, \\
&\frac{\partial U_{m+2}}{\partial z} = \frac{\partial U_{m}}{\partial x}
\end{aligned} \tag{13}$$

for all integer number m.

It is evident that if the functions $U_k : \Omega \to \mathbb{R}$ have continuous second-order partial derivatives in a domain Ω and satisfy the conditions (13), then they satisfy Eq. (2) in Q. Indeed, in this case the function (12) is doubly differentiable in the sense of Gateaux and, therefore, satisfies the equality (5) in the domain Ω_ζ.

In the following theorem we establish that every monogenic function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{G}}$ generates a family of harmonic vectors.

Theorem 3.3. *Every monogenic function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{G}}$ generates harmonic vectors*

$$\mathbf{V} := (U_{2m+2}, U_{2m+1}, U_{2m})$$

in the domain Ω for all integer number m.

Proof. Note that the system (13) can be rewritten in the following equivalent form:

$$\begin{aligned}
&\frac{\partial U_{2m+2}}{\partial x} + \frac{\partial U_{2m+1}}{\partial y} + \frac{\partial U_{2m}}{\partial z} = 0\,, \\
&\frac{\partial U_{2m}}{\partial y} - \frac{\partial U_{2m+1}}{\partial z} = 0\,, \\
&\frac{\partial U_{2m+2}}{\partial z} - \frac{\partial U_{2m}}{\partial x} = 0\,, \\
&\frac{\partial U_{2m+1}}{\partial x} - \frac{\partial U_{2m+2}}{\partial y} = 0
\end{aligned} \tag{14}$$

for all integer number m. Thus, the vector

$$\mathbf{V} := (U_{2m+2}, U_{2m+1}, U_{2m})$$

satisfies the equations (1) in Ω. The theorem is proved.

The following theorem show that all harmonic vectors have a relation to monogenic functions taking values in the topological vector space $\widetilde{\mathbb{G}}$.

Theorem 3.4. *For every function $U_1 : \Omega \to \mathbb{R}$ harmonic in a simply connected domain $\Omega \subset \mathbb{R}^3$ there exist a monogenic function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{G}}$ such that U_1 is a component of the expansion* (12).

Proof. First of all, note that there exists a harmonic vector $\mathbf{V}_0^0 := (v_2^0, U_1, v_0^0)$ in the domain Ω. Moreover, for any vector $\mathbf{V}_0 := (v_2, U_1, v_0)$ harmonic in Ω, the components v_0, v_2 are determined accurate within the real part and the imaginary part of any function $f_0(t)$ holomorphic in the domain $\{t = x + iz : (x, y, z) \in \Omega\}$ of the complex plane, i.e. the equalities

$$v_0(x, y, z) = v_0^0(x, y, z) + \operatorname{Re} f_0(x + iz)\,,$$

$$v_2(x, y, z) = v_2^0(x, y, z) + \operatorname{Im} f_0(x + iz)$$

hold for all $(x, y, z) \in \Omega$. Then, using Theorem 3.3, we find the functions U_0 and U_2, namely: $U_0 := v_0, \quad U_2 := v_2$.

Now, let us show that the conditions (14) allow to determine the functions U_{2m+1}, U_{2m+2} if the function U_{2m} are already determined for all natural m. Indeed, in this case there exists a harmonic vector

$$\mathbf{V}_{2m}^0 := (v_{2m+2}^0, v_{2m+1}^0, U_{2m})$$

in the domain Ω. Moreover, for any vector

$$\mathbf{V}_{2m} := (v_{2m+2}, v_{2m+1}, U_{2m})$$

harmonic in Ω, the components v_{2m+1}, v_{2m+2} are determined accurate within the real part and the imaginary part of any function $f_{2m}(t)$ holomorphic in the domain $\{t = x + iy : (x, y, z) \in \Omega\}$ of the complex plane, i.e. the equalities

$$v_{2m+1}(x, y, z) = v_{2m+1}^0(x, y, z) + \operatorname{Re} f_{2m}(x + iy)\,,$$

$$v_{2m+2}(x, y, z) = v_{2m+2}^0(x, y, z) + \operatorname{Im} f_{2m}(x + iy)$$

hold for all $(x, y, z) \in \Omega$. Then, using Theorem 3.3, we find the functions U_{2m+1} and U_{2m+2}, namely: $U_{2m+1} := v_{2m+1}, \quad U_{2m+2} := v_{2m+2}$.

Finally, let us also show that the conditions (14) allow to determine the functions U_{2m}, U_{2m+1} if the function U_{2m+2} are already determined for all negative integer number m. Indeed, in this case there exists a harmonic vector

$$\mathbf{V}_{2m+1}^0 := (U_{2m+2}, v_{2m+1}^0, v_{2m}^0)$$

in the domain Ω. Moreover, for any vector

$$\mathbf{V}_{2m+1} := (U_{2m+2}, v_{2m+1}, v_{2m})$$

harmonic in Ω, the components v_{2m}, v_{2m+1} are determined accurate within the real part and the imaginary part of any function $f_{2m+1}(t)$ holomorphic in the domain $\{t = z + iy : (x, y, z) \in \Omega\}$ of the complex plane, i.e. the equalities

$$v_{2m}(x, y, z) = v_{2m}^0(x, y, z) + \operatorname{Im} f_{2m+1}(z + iy)\,,$$

$$v_{2m+1}(x, y, z) = v_{2m+1}^0(x, y, z) + \operatorname{Re} f_{2m+1}(z + iy)$$

hold for all $(x, y, z) \in \Omega$. Then, using Theorem 3.3, we find the functions U_{2m} and U_{2m+1}, namely: $U_{2m} := v_{2m}, \quad U_{2m+1} := v_{2m+1}$.

Thus, the functions U_k obtained in such a way satisfy the system (14) and form the function (12) monogenic in Ω_ζ. The theorem is proved.

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CHARAKTERYZACJA PRZESTRZENNYCH PÓL POTENCJALNYCH PRZEZ FUNKCJE MONOGENICZNE W PRZESTRZENIACH NIESKOŃCZENIE WYMIAROWYCH Z PRZEMIENNYM MNOŻENIEM

S t r e s z c z e n i e

Rozważamy funkcje monogeniczne o wartościach w topologicznych przestrzeniach wektorowych będących rozszerzeniem pewnych nieskończenie wymiarowych przemiennych algebr Banacha stowarzyszonych z trójwymiarowym równaniem Laplace'a. Wykazujemy, że każda funkcja harmoniczna jest składową wspomnianych funkcji monogenicznych.

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In memory of
Professor Promarz M. Tamrazov

Vladimir M. Miklyukov

ON LOCAL APPROXIMATION OF FUNCTIONS IN ANISOTROPIC SPACES

Summary

We prove a generalization of the theorem on existence of total differential at a point for functions in domains of anisotropic spaces. As corollaries we obtain some results on approximation of functions by n-polynomials close to inner and boundary points in subdomains of $\mathbb{R}^n$.

Keywords and phrases: approximation, anisotropic space, total differential, polynomial, boundary point

1. Anisotropic metric

Let $\mathcal{X}$ be a nonempty set and let $r : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ – be a function satisfying the following properties:

$$\begin{aligned} &\alpha) \quad r(x,x) &&= 0 \text{ and } r(x,y) \geq 0 \text{ for all } x,y \in \mathcal{X}; \\ &\beta) \quad r(x,y) &&\leq r(x,z) + r(z,y) \text{ for all } x,y,z \in \mathcal{X}. \end{aligned}$$

The pair $(\mathcal{X}, r)$ is called *anisotropic space,* and the function r *anisotropic metric.* Note that we do not assume here the symmetry of the pseudometric r, i.e. in general $r(x,y) \neq r(y,x)$.

Special cases of anisotropic spaces are pseudometric and metric spaces (see, for example, [1, §21], [2]).

For other examples of anisotropic spaces arising in the theory of abstract surfaces see, for example, in [3, Ch. 1].

Let $a \in \mathcal{X}$ and $\varepsilon > 0$ be a real number. Define ε-neighbourhood of a putting

$$O(a, \varepsilon) = \{x \in \mathcal{X} : \rho(a, x) < \varepsilon\}$$

and by standard way we define the basis topological concepts for anisotropic spaces.

2. Ends of domains

Let $\mathcal{X}$ be an anisotropic space and $D \subset \mathcal{X}$ be a domain.

We define ends of D using an analogy with the theory of Carathéodory prime ends (see, for example, [4, §3]). Below we follow [5, Sec. 5.1.2].

For an arbitrary set $U \subset D$ we put $[U] = \overline{U} \setminus \partial D$, where $\overline{U}$ is closure with respect to D. Let $\{U_k\}$, $k = 1, 2, \dots$ be a sequence of subdomains $U_k \subset D$ with properties:

(i) for all $k = 1, 2, \dots \quad [U_{k+1}] \subset U_k$,

(ii) $\bigcap\limits_{k=1}^{\infty} [U_k] = \emptyset$.

We will call *chain* of subdomains in D an arbitray sequence $\{U_k\}$ with above described properties (i) and (ii).

Let $\{U'_k\}$, $\{U''_k\}$ be two chains of subdomains in D. We say that the chain U'_k is *contained* in the chain $\{U''_k\}$, if for every $m \geq 1$ there exists a number $k(m)$ such that for all $k > k(m)$ the following property holds $U'_k \subset U''_m$. Two chains each of which is containing in other are called *equivalent*. The classes of equivalency ξ of chains are called *ends* of the domain D.

In order to define an end ξ it is enough to give at least one representative in the class of equivalence. If an end ξ is defined by the chain $\{U_k\}$, then we write $\xi \asymp \{U_k\}$.

The *body* of an end $\xi \asymp \{U_k\}$ is the following set

$$|\xi| = \cap_{k=1}^{\infty} \overline{U_k} \, .$$

It is clear that this set is independent of the choice of the chain $\{U_k\}$.

Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of points $x_m \in D$ which does not have limit points in D. Such sequences are known as *nonconvergent* in D.

Let $a_\xi \in |\xi|$ be an arbitrary point. A nonconvergent in D sequence of points $x_k \in D$ *converges to a point* a_ξ *with respect to the topology of* ξ, if $x_k \to a_\xi$ (with respect to the topology of $\mathcal{X}$) and for some chain $\{U_k\} \in \xi$ the following property holds: for every $k = 1, 2, \dots$ there exists a number $m(k)$ such that $x_m \in U_k$ for all $m > m(k)$.

Let D be a domain in $\mathcal{X}$, ξ be an end of D, and $a_\xi \in |\xi|$ be a point. We say that a subdomain D' of D *adjoins* at the point a_ξ, if $a_\xi \in \partial D'$ and every sequence $x_k \in D'$ converging at a_ξ with respect to the topology of $\mathcal{X}$ converges at this point with respect to the topology of ξ.

We say that a function $f : D \to \mathbb{R}^1$ satisfies the relation

$$\lim_{x \to a_\xi} f(x) = A\,,$$

if $f(x_k) \to A$ as $x_k \to a_\xi$ along every sequence $x_k \in D$ which converges at a_ξ with respect to the topology of ξ. The quantity A is denoted in this case by the symbol $f(a_\xi)$.

Let y and a be an arbitrary pair of points such that $y \in D$ and a is either an inner point of D or $a = a_\xi \in |\xi|$, where ξ is an end of the domain $D \subset \mathcal{X}$. We say that a simple Jordan arc γ, given by a parametrization $x(\tau) : [0, 1) \to D$, *leads from the point* y *to the point* a, if $x(0) = y$ and

$$\lim_{\tau \to 1} x(\tau) = a \quad \text{as} \quad a \in D$$

and there exists a sequence $\tau_k \to 1$, along which

$$\lim_{\tau_k \to 1} x(\tau_k) = a_\xi \quad \text{as} \quad a \in |\xi|\,.$$

3. Intrinsic anisotropic distance

Let $D \subset \mathcal{X}$ be a domain. Let δ_D be the intrinsic anisotropic distance in the domain D, i.e.

$$\delta_D(x', x'') = \inf \operatorname{length} \gamma\,, \quad x',\, x'' \in D,$$

where the infimum is taken over all oriented, locally rectifiable arcs $\gamma \subset D$, leading from point x' to point x''.

Let D be a domain in an anisotropic space $\mathcal{X}$ with a metric r. Let $a \in \overline{D}$. In addition, if $a \in \partial D$, then we assume that $a = a_\xi \in |\xi|$, where ξ is an end of the domain D.

For an arbitrary $t > 0$, we put

$$B^-_D(a, t) = \{x \in D : \delta_D(a, x) < t\}\,, \qquad B^+_D(a, t) = \{x \in D : \delta_D(x, a) < t\}\,,$$

and for a function $f : D \setminus \{a\} \to \mathbb{R}^1$ let

$$\lim_{x \to a}{}^{-} f(x) = \lim_{\substack{\delta_D(a,x) \to 0 \\ x \in D}} f(x),$$

$$\lim_{x \to a}{}^{+} f(x) = \lim_{\substack{\delta_D(x,a) \to 0 \\ x \in D}} f(x).$$

Note that in general

$$\lim_{x \to a}{}^{-} f(x) \neq \lim_{x \to a}{}^{+} f(x).$$

4. Admissible domains

Let $\mathcal{X}$ be a space with an anisotropic metric r and D be a domain in $\mathcal{X}$. We call the domain D *admissible*, if

1) for an arbitrary pair of points x', x'' in D the intrinsic distance $\delta_D(x', x'') < \infty$, and

2) above property 1) holds for any subdomain $D' \subset D$,

Simplest examples of domains without the property 1) are open arcs lying in $\mathbb{R}^2$ and not having rectifiable subarcs.

Examples of domains with the property 1) however without the property 2), we can easilay construct if we take as $\mathcal{X}$ some nonintersection discs connected with one-dimensional rectifiable arcs.

Lemma 4.1. *If a domain $D \subset \mathcal{X}$ is admissible then for any pairs of points $a', a'' \in \overline{D}$ there exists an open locally rectifiable arc $\gamma \subset D$, leading from a' to a''.*

Proof. By virtue of the assumption 1) for the domain D, it is enough to consider the case, in which $a' \in D$ and a'' belongs to the body $|\xi|$ of some end ξ of D.

Denote by $\Delta_D(t, a'')$ a connected component of the set $B^-_D(a'', t)$. It is clear that $\Delta_D(t, a'')$ adjoins at the end ξ.

Fix a point $x_1 \in \Delta_D(1, a'')$, different from a'. By the assumption 1), there exists an Jordan rectifiable arc $\gamma_1 \subset D$, leading from a' to x_1.

Fix $x_2 \in \Delta_D(\frac{1}{2}, a'')$. By virtue of the assumption 2) for $\Delta_D(1, a'')$, there exists an Jordan rectifiable arc γ_2', leading from x_1 to x_2. The arc $\gamma_2 = \gamma_1 \cup \gamma_2'$ is rectifiable and leads from a' to x_2.

Fix $x_3 \in \Delta_D(\frac{1}{3}, a'')$, different from a', x_1 and x_2. Using 2) for the domain $\Delta_D(\frac{1}{2}, a'')$, we find an Jordan rectifiable arc γ_3', leading from x_2 to x_3. The arc $\gamma_3 = \gamma_1 \cup \gamma_2' \cup \gamma_3'$ is rectifiable and leads from a' to x_3.

If we continue this process indefinitely, then we obtain the arc

$$\gamma = \gamma_1 \cup {\gamma'}_2 \cup \ldots \cup {\gamma'}_n \cup \ldots ,$$

where the arcs ${\gamma'}_n$ are rectifiable, connect points x_{n-1}, x_n and are situated in $\Delta_D(\frac{1}{n-1}, a'')$. The arc γ is desired. □

Remark 4.1. From the proof of the lemma, it follows that if $a', a'' \in \overline{D}$ and $\delta_D(a', a'') < \infty$, then the arc $\gamma \subset D$ can be chosen such that its length is any amount close to $\delta_D(a', a'')$.

5. Setting of the problem

Let D be a domain in an anisotropic space $\mathcal{X}$ with a metric r. Let $a \in \overline{D}$ be a fixed point such that

$$(1) \qquad \delta^{\pm}_{a,D}(x) < \infty \quad \forall x \in D,$$

where

$$\delta^{\pm}_{a,D}(x) = \begin{cases} \delta^{-}_{a,D}(x) & = \delta_D(a, x), \\ \delta^{+}_{a,D}(x) & = \delta_D(x, a). \end{cases}$$

Moreover if $a \in \partial D$, we assume that some end ξ of D is given and $a \in |\xi|$. Here and below the notation $\delta^{\pm}_{a,D}(x) < \infty$ means one of two conditions: either $\delta^{+}_{a,D}(x) < \infty$ or $\delta^{-}_{a,D}(x) < \infty$. Thus by (1) we denote two different relations.

Consider two classes $\varsigma^{\pm}(D,a)$ of functions $\varsigma : D \to \mathbb{R}^1$ with corresponding properties:

$$\lim\nolimits^{\pm}_{a' \to a} \varsigma(a') = 0 \,.$$

We call such functions by *simplest.*

Let $f : D \backslash \{a\} \to \mathbb{R}^1$ be a function. Our goal is to explain conditions under which there exist functions $\varsigma \in \varsigma^{\pm}(D,a)$ and

$$\varepsilon : D \setminus \{a\} \to \mathbb{R}^1, \quad \lim\nolimits^{\pm}_{a' \to a} \varepsilon(a') = 0, \tag{2}$$

such that, respectively,

$$f(a') - f(a) = \varsigma(a') + \varepsilon(a')\, \delta^{\pm}_{a,D}(a') \,, \quad a' \in D. \tag{3}$$

6. Inclination of a graph

Let $\mathcal{X}$ be an anisotropic space $\mathcal{X}$ with a metric r. Let D be a domain in $\mathcal{X}$, and $f : D \to \mathbb{R}^1$ be a function. For an arbitrary point $x \in D$ we put

$$\lambda_+(x,f) = \sup_{\gamma \ni x} \overline{\lim}_{\substack{y \to x \\ y \in \gamma}} \frac{|f(y) - f(x)|}{r(y,x)} \,,$$

where the supremum is taken over all locally rectifiable arcs $\gamma \subset D$, $\gamma \ni x$.

The quantity $\lambda_+(x,f)$ characterizes an *inclination* of the graph f at $x \in \mathcal{X}$ to $\mathcal{X}$. In the case, D is a subdomain of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^1$ is a C^1-function, we have

$$\lambda_+(x,f) = |\nabla f(x)| \,.$$

Below we will need

Lemma 6.1. *If a function $f : D \subset \mathcal{X} \to \mathbb{R}^1$ is absolutely continuous along an oriented locally rectifiable arc $\gamma \subset D$, leading from a point a to a point b, then*

$$|f(b) - f(a)| \le \left| \int_\gamma \lambda_+(x,f)\, ds_\gamma \right| . \tag{4}$$

The *proof* is exactly as in [4, §2, Ch. I] for $D \subset \mathbb{R}^n$.

7. Main theorem

The following statement is true:

Theorem 7.1. *Let $D \subset \mathcal{X}$ be an admissible domain, $a \in \overline{D}$ be a fixed point satisfying to* (1)*, and $g : D \to \mathbb{R}^1$ be a function with the properties:*

$$\lambda_+(x,g) < \infty \quad \forall x \in D \quad \textit{and} \quad \lim_{\delta^{\pm}_{a,D}(x) \to 0} \lambda_+(x,g) = 0. \tag{5}$$

Then there exists $g(a) = \lim_{\delta^{\pm}_{a,D}(x)\to 0} g(x)$ *such that*

$$g(a') = g(a) + \delta^{\pm}_{a,D}(a')\, \varepsilon_a(a'), \quad \textit{where} \quad \lim_{\delta^{\pm}_{a,D}(a')\to 0} \varepsilon_a(a') = 0. \tag{6}$$

Moreover, we can put

$$\varepsilon_a(a') = \sup_{x \in B^{\pm}_D(a,t)} \lambda_+(x,g)\,, \quad t = \delta^{\pm}_{a,D}(a'). \tag{7}$$

Proof. At first, we note that the finitness in D of the quantity $\lambda_+(x,g)$ implies continuity of g. Let $R > 0$ and $s > 0$ be such that

$$\lambda_+(x,g) < s \quad \forall x \in B^{\pm}_D(a,R)\,.$$

The set $B^{\pm}_D(a,R)$ is open and adjoins to the boundary point a. Take an arbitrarily a point $b \in B^{\pm}_D(a,R)$. Since D is admissible then by Lemma 4.1 there exists a locally rectifiable arc $\gamma \subset B^{\pm}_D(a,R)$ leading from b to a.

Everywhere along γ, the property $\lambda_+(x,g) < s$ holds. Thus, the function g is absolutely continuous along γ, and by Lemma 6.1 for an arbitrary pair of points $x', x'' \in \gamma$, we have

$$|g(x'') - g(x')| \le \int\limits_{\gamma(x',x'')} \lambda_+(x,g)\, ds_\gamma. \tag{8}$$

From here, in particular, it follows the existence of $\lim g(x)$ as $x \to a$ along γ. Moreover, it is clear that this limit does not depend on $\gamma \subset B^{\pm}_D(a,R)$ and there exists

$$g(a_\xi) \equiv \lim_{\substack{x\to a \\ x \in B^{\pm}_D(a,s)}} g(x)\,.$$

By (8) we conclude that for every point $a' \in \gamma$ the following property holds

$$|g(a') - g(a_\xi)| \le \int\limits_{\gamma(a,a')} \lambda_+(x,g)\, ds_\gamma \le$$

$$\le \operatorname{length} \gamma(a,a') \sup_{x \in B^{\pm}_D(a,R)} \lambda_+(x,g).$$

Using the Remark 4 we see that the arc $\gamma(a,a')$ can be choosen such that length $\gamma(a,a')$ will be any amount close to $\delta^{\pm}_{a,D}(a')$. Because the quantity

$$\sup_{x \in B^{\pm}_D(a,R)} \lambda_+(x,g) \to 0, \quad (R \to 0),$$

the relations (6) and (7) hold. □

Putting

$$g(x) = f(x) - f(a) - \varsigma(x), \quad \varsigma(x) \in \varsigma^{\pm}(D, a),$$

we obtain

Corollary 7.1. *If a domain D is admissible and satisfies to* (1), *and a function f is such that for g the assumptions* (5) *hold, then f satisfies to* (3).

8. Applications for functions on subdomains of $\mathbb{R}^n$

8.1. Cases conserning inner points

At first we denote the following well-known statement.

Theorem 8.1. *Let D be a domain in the Euclidean space $\mathbb{R}^n$ and $a \in D$. If $f : D \to \mathbb{R}^1$ is a function having the partial derivatives $\partial f/\partial x_i$ $(i = 1, 2, \ldots, n)$ at every point $y \in D$ and these derivatives are continuous at a, then f has a total differential at the point a.*

Proof. Put in Theorem 7.1

$$g(x) = f(x) - f(a) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\,(x_i - a_i), \quad a = (a_1, \ldots, a_i, \ldots, a_n).$$

We have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a).$$

Because

$$g(a + \Delta x) - g(a) = f(a + \Delta x) - f(a) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\,\Delta x_i,$$

where

$$\Delta x = (\Delta x_1, \ldots, \Delta x_i, \ldots, \Delta x_n),$$

it then follows by the mean value theorem,

$$g(a + \Delta x) - g(a) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\xi_i) - \frac{\partial f}{\partial x_i}(a) \right) \Delta x_i\,.$$

Here ξ_i are some points such that $|\xi_i - a| \le |\Delta x|$. Thus, for sufficiently small Δx, the quantity $\lambda_+(x, g)$ satisfies to (5). The necessary statement follows from Theorem 7.1. □

Now we consider other known statement on existence of the total differential of order m at a point.

Theorem 8.2. *Let $m, n \ge 1$ be integers, let D be a subdomain of the Euclidean space $\mathbb{R}^n$ and $a = (a_1, \ldots, a_n) \in D$. If $f : D \to \mathbb{R}^1$ is a function having on D the continuous partial derivatives of the orders k, $1 \le k \le m$,*

$$\frac{\partial^k f}{\partial x_{i_1}...\partial x_{i_k}}(x), \quad 1 \le i_1, \dots, i_k \le n,$$

and the continuous at the point a partial derivatives of the order $m+1$:

$$\frac{\partial^{m+1} f}{\partial x_{i_1}...\partial x_{i_{m+1}}}(y), \quad 1 \le i_1, \dots, i_{m+1} \le n,$$

then

$$f(x) = f(a) + \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_{i_1}}(a)(x_{i_1} - a_{i_1}) +$$

$$+\frac{1}{2!} \sum_{1 \le i_1, i_2 \le n} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(a)(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) + \dots$$

$$+\frac{1}{(m+1)!} \sum_{1 \le i_1, \dots, i_{m+1} \le n} \frac{\partial^{m+1} f}{\partial x_{i_1}...\partial x_{i_{m+1}}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_{m+1}} - a_{i_{m+1}}) +$$

$$+o(|x-a|^{m+1}), \quad x \to a.$$

Proof. We use Theorem 7.1. We put

$$g(x) = f(x) - f(a) - \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_{i_1}}(a)\,(x_{i_1} - a_{i_1})$$

$$-\frac{1}{2!} \sum_{1 \le i_1, i_2 \le n} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(a)(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) -$$

$$-\frac{1}{(m+1)!} \sum_{1 \le i_1, \dots, i_{m+1} \le n} \frac{\partial^{m+1} f}{\partial x_{i_1}...\partial x_{i_{m+1}}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_{m+1}} - a_{i_{m+1}}).$$

Here

$$\lambda_+(x, g) = |\nabla g(x)|\,.$$

Let $m = 1$. We have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a) - \frac{1}{2}\frac{\partial}{\partial x_i}\left[\sum_{p=1}^{n}(x_p - a_p)\sum_{q=1}^{n}\frac{\partial^2 f}{\partial x_p \partial x_q}(a)(x_q - a_q)\right].$$

We find

$$\left[\sum_{p=1}^{n}(x_p - a_p)\sum_{q=1}^{n}\frac{\partial^2 f}{\partial x_p \partial x_q}(a)(x_q - a_q)\right]'_{x_i} =$$

$$= \sum_{p=1}^{n}(x_p - a_p)'_{x_i}\sum_{q=1}^{n}\frac{\partial^2 f}{\partial x_p \partial x_q}(a)(x_q - a_q) +$$

$$+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\partial^2 f}{\partial x_p \partial x_q}(a)(x_p - a_p)(x_q - a_q)'_{x_i} =$$

$$= \sum_{q=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_q}(a)(x_q - a_q) + \sum_{p=1}^{n}(x_p - a_p)\frac{\partial^2 f}{\partial x_p \partial x_i}(a) =$$

$$= 2\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_j - a_j).$$

Thus, using the mean value theorem, for an arbitrary $i = 1, 2, \ldots, n$ we have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a) - \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_j - a_j) =$$

$$= \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_j)(x_j - a_j) - \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_j - a_j) =$$

$$= \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_{ji}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)(x_j - a_j),$$

where ξ_{ji} are some points of the set $B(a, |x - a|)$.

Hence it is clear that quantities $\lambda_+(x, g)$ satisfy (5).

The necessary statement follows from Theorem 7.1, because the quantity

$$\varepsilon_a(x) = o(|x - a|^2) \quad \text{as} \quad x \to a.$$

In the case $m = 2$ it is enough to remark in addition only, that

$$\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a) - \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_j - a_j) =$$

$$= \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_{ij}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)(x_j - a_j) =$$

$$= \sum_{j,k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\xi_{ij\,k})(x_j - a_j)(x_k - a_k), \quad \xi_{ij\,k} \in B(a, |x - a|).$$

In the case $m > 2$ the arguments are analogous. □

8.2. Cases concerning boundary points

Below we bring boundary versions of Theorem 8.2. See also some boundary versions of Theorem 8.1 for weight Sobolev classes in [5, Sect. 5.6-5.8].

Theorem 8.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 1$, let ξ be an end of D and $a = (a_1, \ldots, a_n) \in |\xi|$ be a boundary point satisfying* (1) *with respect to the intrinsic distance $\delta_D(x, y)$ on D.*

Let $f : D \to \mathbb{R}^1$ be a function having on D the continuous partial derivatives:

$$\frac{\partial f}{\partial x_{i_1}}(x), \quad \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x), \quad 1 \leq i_1, i_2 \leq n.$$

Suppose that the following finite limits exist

$$A = A(a) = \lim f(x), \quad A_k = A_k(a) = \lim \frac{\partial f}{\partial x_k}(x), \quad k = 1, \dots, n,$$

$$A_{i_1\,i_2} = A_{i_1\,i_2}(a) = \lim \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x), \quad 1 \le i_1,\, i_2 \le n,$$

as $\delta_D(x,a) \to 0$.

Then

$$f(x) = A + \sum_{i_1=1}^{n} A_{i_1}(x_{i_1} - a_{i_1}) + O(\delta_D^2(x,a)) \quad (\delta_D(x,a) \to 0).$$

Proof. We use Theorem 7.1. We put

$$g(x) = f(x) - A - \sum_{i_1=1}^{n} A_{i_1}\,(x_{i_1} - a_{i_1}) - \frac{1}{2} \sum_{1 \le i_1, i_2 \le n} A_{i_1 i_2}(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}).$$

As above, we remark that

$$\lambda_+(x,g) = |\nabla g(x)|\,.$$

We have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) - A_i - \left[\sum_{p=1}^{n}(x_p - a_p) \sum_{q=1}^{n} A_{p\,q}(x_q - a_q)\right]'_{x_i} \quad (1 \le i \le n). \tag{9}$$

Next we find

$$\left[\sum_{p=1}^{n}(x_p - a_p) \sum_{q=1}^{n} A_{p\,q}(x_q - a_q)\right]'_{x_i} = \tag{10}$$

$$= \sum_{p=1}^{n}(x_p - a_p)'_{x_i} \sum_{q=1}^{n} A_{p\,q}\,(x_q - a_q) + \sum_{p=1}^{n}\sum_{q=1}^{n} A_{p\,q}\,(x_p - a_p)(x_q - a_q)'_{x_i}$$

$$= 2\sum_{q=1}^{n} A_{i\,q}(x_q - a_q).$$

Thus, for arbitrary $i = 1, 2, \dots, n$ the following property holds

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) - A_i - \sum_{j=1}^{n} A_{i\,j}(a)(x_j - a_j).$$

The domain D is admissible and the boundary point $a \in |\xi|$ is accessible outwards D by an arc of finite length. Fix $x \in D$ and a path $\gamma \subset D$, leading from a to x, infinite smooth and such that

$$\mathrm{l}(\gamma) \le 2\delta_D(a,x), \quad \mathrm{l}(\gamma) = \operatorname{length}\gamma. \tag{11}$$

Let

$$x(s) : (0, \mathrm{l}(\gamma)) \to D, \quad x(s) \in C^\infty, \quad \lim_{s\to 0} x(s) = a,\ \lim_{s \to \mathrm{l}(\gamma)} x(s) = x,$$

be a natural parametrization of γ.

Using the mean value theorem, we have

$$\frac{\partial f}{\partial x_i}(x) - A_i = \left(\frac{\partial f}{\partial x_i}(x(s))\right)'_{s=\eta} \mathrm{l}(\gamma), \quad \eta \in [0, \mathrm{l}(\gamma)),$$

and

$$\frac{\partial f}{\partial x_i}(x) - A_i = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x(s_i)) x_j'(s_i)\, \mathrm{l}(\gamma), \quad |x'(s_i)| = 1,$$

where $s_i \in (0, \mathrm{l}(\gamma))$ are some points.

The points $\xi_i = x(s_i) \in \gamma \subset B_D(a, |x-a|)$. Thus,

$$\frac{\partial f}{\partial x_i}(x) - A_i = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_i)\, x_j'(s_i)\, \mathrm{l}(\gamma)$$

and by virtue of (11),

$$\left|\frac{\partial f}{\partial x_i}(x) - A_i\right| \leq 2 \sum_{j=1}^{n} \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_i)\right| \delta_D(a, x).$$

Remark that

$$\left|\sum_{j=1}^{n} A_{i\,j}(x_j - a_j)\right| \leq \left(\sum_{j=1}^{n} A_{i\,j}^2\right)^{\frac{1}{2}} |x-a| \leq \left(\sum_{j=1}^{n} A_{i\,j}^2\right)^{\frac{1}{2}} \delta_D(a, x).$$

Moreover, using (9), we find

$$\lambda_+^2(x, g) = |\nabla g(x)|^2 \leq 2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(x) - A_i\right)^2 + 2 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij}(a)(x_j - a_j)\right)^2 \leq$$

$$\leq 8\delta_D^2(a, x) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_i)\right|\right)^2 + 2|x-a|^2 \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2(a)$$

and the quantity $\lambda_+(x, g)$ satisfies (5).

Using Theorem 7.1 we have

$$f(x) = A + \sum_{i_1=1}^{n} A_{i_1}(x_{i_1} - a_{i_1}) + \frac{1}{2} \sum_{1 \leq i_1, i_2 \leq n} A_{i_1 i_2}(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) + \delta_D(a, x)\varepsilon_a(x),$$

where

$$\varepsilon_a(x) = O(\delta_D(a, x)) \quad \text{as} \quad \delta_D(a, x) \to 0.$$

Then

$$\left|f(x) - A - \sum_{i_1=1}^{n} A_{i_1}(x_{i_1} - a_{i_1})\right| \leq \frac{1}{2}|x-a|^2 \sum_{1 \leq i_1, i_2 \leq n} |A_{i_1 i_2}| + \delta_D(a, x)\varepsilon_a(x) =$$

$$= O(\delta_D^2(a, x)) \qquad (\delta_D(a, x) \to 0)$$

and the necessary statement is true. □

Let D be a subdomain of $\mathbb{R}^n$, $n \geq 1$, let ξ be an end of D and let $a = (a_1, \dots, a_n) \in |\xi|$ be a boundary point, satisfying (1) with respect to the intrinsic distance $\delta_D(x,y)$ on D. We will say that the point a satisfies the *α-condition*, $0 < \alpha \leq 1$, if there exist constants $R > 0$ and $0 < \mathcal{A} < \infty$ such that for an arbitrary point $x \in B_D(a,R)$ there exists a C^1-path $\gamma \subset B_D(a,R)$, leading from a to x and having the tangent vectors

$$\overline{e}(y), \quad |\overline{e}(y)| = 1, \quad \forall\, y \in \gamma,$$

with

$$\sup_{y\in\gamma} |\overline{e}(y)\, \mathrm{l}(\gamma) - (y-a)| \leq \mathcal{A}\, \delta_D^{1+\alpha}(x,a). \tag{12}$$

It is clear that the condition (12) is invariant with respect to the bi-Lipschitz homeomorphisms $f : D \to \mathbb{R}^n$.

Theorem 8.4. *Let D be a subdomain of $\mathbb{R}^n$, $n \geq 1$, let ξ be an end of D and let $a = (a_1, \dots, a_n) \in |\xi|$ be a point satisfying* (12).

Let $f : D \to \mathbb{R}^1$ be a function having on D continuous partial derivatives of the orders k, $0 \leq k \leq m+1$, moreover there exist the limits (*with respect to the topology of the end ξ*)

$$\lim_{x\to a} \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(x) = 0, \quad 1 \leq i_1, \dots, i_k \leq n, \quad 2 \leq k \leq m, \tag{13}$$

and the finite limits

$$A_{i_1 \dots i_{m+1}} = \lim_{x\to a} \frac{\partial^{m+1} f}{\partial x_{i_1} \dots \partial x_{i_{m+1}}}(x), \quad 1 \leq i_1, \dots, i_{m+1} \leq n. \tag{14}$$

Then

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) +$$

$$+ \frac{1}{(m+1)!} \sum_{1 \leq i_1, \dots, i_{m+1} \leq n} A_{i_1 \dots i_{m+1}} (x_{i_1} - a_{i_1}) \dots (x_{i_{m+1}} - a_{i_{m+1}}) +$$

$$+ O(\delta_D^{(m+1)(1+\alpha)}(x,a))\, \chi_a(x) \quad (\delta_D(x,a) \to 0),$$

where

$$\chi_a(x) \to 0 \quad \textit{as} \quad \delta_D(x,a) \to 0.$$

If except (14) *the assumptions are true, and derivatives*

$$\frac{\partial^{m+1} f}{\partial x_{i_1} \dots \partial x_{i_{m+1}}}(x), \quad 1 \leq i_1, \dots, i_{m+1} \leq n,$$

are bounded near ξ, then

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) +$$

$$+\frac{1}{(m+1)!}\sum_{1\le i_1,\dots,i_{m+1}\le n} A_{i_1\dots i_{m+1}}(x_{i_1}-a_{i_1})\dots(x_{i_{m+1}}-a_{i_{m+1}})+$$

$$+O(\delta_D^{(m+1)(1+\alpha)}(x,a))\,\chi_a(x)\quad(\delta_D(x,a)\to 0),$$

where

$$\chi_a(x)=O(1)\quad as\quad \delta_D(x,a)\to 0.$$

Proof. Let

$$g(x)=f(x)-f(a)-\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)(x_i-a_i)-$$

$$-\frac{1}{(m+1)!}\sum_{1\le i_1,\dots,i_{m+1}\le n} A_{i_1\dots i_{m+1}}(x_{i_1}-a_{i_1})\dots(x_{i_{m+1}}-a_{i_{m+1}}).$$

We have

$$\frac{\partial g}{\partial x_i}(x)=\frac{\partial f}{\partial x_i}(x)-\frac{\partial f}{\partial x_i}(a)$$

$$-\frac{1}{(m+1)!}\left[\sum_{1\le i_1,\dots,i_{m+1}\le n} A_{i_1\dots i_{m+1}}(x_{i_1}-a_{i_1})\dots(x_{i_{m+1}}-a_{i_{m+1}})\right]'_{x_i}$$

and

$$\left[\sum_{1\le i_1,\dots,i_{m+1}\le n} A_{i_1\dots i_{m+1}}(x_{i_1}-a_{i_1})\dots(x_{i_{m+1}}-a_{i_{m+1}})\right]'_{x_i}=$$

$$=\sum_{1\le j_1,\dots,j_m\le n} A_{ij_1\dots j_m}(x_{j_1}-a_{j_1})\dots(x_{j_m}-a_{j_m})+$$

$$+(x_i-a_i)\left[\sum_{1\le j_1,\dots,j_m\le n} A_{ij_1\dots j_m}(x_{j_1}-a_{j_1})\dots(x_{j_m}-a_{j_m})\right]'_{x_i}=$$

$$=\sum_{1\le j_1,\dots,j_m\le n} A_{ij_1\dots j_m}(x_{j_1}-a_{j_1})\dots(x_{j_m}-a_{j_m})+$$

$$+(x_i-a_i)\sum_{1\le k_1,\dots,k_{m-1}\le n} A_{iik_1\dots k_{m-1}}(x_{k_1}-a_{k_1})\dots(x_{k_{m-1}}-a_{k_{m-1}})+\dots$$

Thus,

$$\frac{\partial g}{\partial x_i}(x)=\frac{\partial f}{\partial x_i}(x)-\frac{\partial f}{\partial x_i}(a)$$

$$-\frac{1}{(m+1)!}\sum_{1\le j_1\le\dots\le j_m\le n} A_{ij_1\dots j_m}(x_{j_1}-a_{j_1})\dots(x_{j_m}-a_{j_m})-$$

$$-\frac{x_i-a_i}{(m+1)!}\sum_{1\le k_1,\dots,k_{m-1}\le n} A_{iik_1\dots k_{m-1}}(x_{k_1}-a_{k_1})\dots(x_{k_{m-1}}-a_{k_{m-1}})-\dots$$

From here, $\lambda_+(x,g) = |\nabla g(x)|$ and (5), (6) are true.

Fix an C^1-arc γ with described properties. Introduce its natural parametrization

$$x(s) : (0, l(\gamma)) \to \gamma, \quad x(0) = a,\ x(l(\gamma)) = x.$$

By the mean value theorem we find $s_1 \in (0, l(\gamma))$ such that

$$f(x) - f(a) = f(x(l(\gamma))) - f(x(0)) = \frac{df}{ds}(x(s_1))l(\gamma) =$$

$$= \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_{i_1}}(x(s_1))x'_{i_1}(s_1)l(\gamma) =$$

$$= \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_{i_1}}(a)x'_{i_1}l(\gamma) + \sum_{i_1=1}^{n} \left(\frac{\partial f}{\partial x_{i_1}}(x(s_1)) - \frac{\partial f}{\partial x_{i_1}}(a) \right) x'_{i_1}(s_1)l(\gamma).$$

Now, using (13) and the mean value theorem we have

$$\frac{\partial f}{\partial x_{i_1}}(\xi_1) - \frac{\partial f}{\partial x_{i_1}}(a) = \sum_{i_2=1}^{n} \frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}(x(s_2))x'_{i_2}(s_2)l(\gamma), \quad \xi_1 = x(s_1), \quad s_2 \in (0, s_1),$$

and

$$f(x) = f(a) + \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_{i_1}}(a)x'_{i_1}(s_1)l(\gamma) +$$

$$+ \sum_{i_2=1}^{n}\sum_{i_1=1}^{n} \frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}(\xi_2)x'_{i_1}(s_1)x'_{i_2}(s_2)l^2(\gamma), \quad \xi_2 = x(s_2), \quad 0 < s_2 < s_1 < l(\gamma).$$

Therefore, after m steps we obtain

$$f(x) = f(a) + \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_i}(a)x'_{i_1}(s_1)l(\gamma) +$$

$$+ \sum_{i_{m+1}=1}^{n} \dots \sum_{i_1=1}^{n} \frac{\partial^{m+1} f}{\partial x_{i_1}\dots\partial x_{i_{m+1}}}(\xi_{m+1})x'_{i_1}(s_1)\dots x'_{i_{m+1}}(s_{m+1})l^{m+1}(\gamma),$$

where

$$\xi_{m+1} = x(s_{m+1}), \quad 0 < s_{m+1} < \dots < s_1 < l(\gamma).$$

From here

$$f(x) = f(a) + \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_i}(a)x'_{i_1}(s_1)l(\gamma) +$$

$$+ \sum_{i_{m+1}=1}^{n} \dots \sum_{i_1=1}^{n} A_{i_1,\dots,i_{m+1}} x'_{i_1}(s_1)\dots x'_{i_{m+1}}(s_{m+1})l^{m+1}(\gamma) +$$

$$+ \sum_{i_1,\dots i_{m+1}} \left(\frac{\partial^{m+1} f}{\partial x_{i_1}\dots\partial x_{i_{m+1}}}(\xi_{m+1}) - A_{i_1,\dots,i_{m+1}} \right) x'_{i_1}(s_1)\dots x'_{i_{m+1}}(s_{m+1})l^{m+1}(\gamma) =$$

$$= \sum_{i_1,\dots i_{m+1}} A_{i_1,\dots,i_{m+1}} x'_{i_1}(s_1)\dots x'_{i_{m+1}}(s_{m+1})l^{m+1}(\gamma) + l^{m+1}(\gamma)\, o(l(\gamma)), \quad l(\gamma) \to 0.$$

Now we use α-condition. We note that from (12) for all $i = 1, \dots , n$ it follows

$$|x_i'\, l(\gamma) - (x_i - a_i)| \le \mathcal{A}\, \delta_D^{1+\alpha}(x,a), \quad x = (x_1, \dots , x_n), \quad a = (a_1, \dots , a_n).$$

Thus,

$$\sum_{1\le i_1,\dots i_{m+1}\le n} A_{i_1,\dots ,i_{m+1}} x_{i_1}'(s_1)\dots x_{i_{m+1}}'(s_{m+1}) l^{m+1}(\gamma) =$$

$$= \sum_{1\le i_1,\dots i_{m+1}\le n} A_{i_1,\dots ,i_{m+1}} (x_{i_1} - a_{i_1})\dots (x_{i_{m+1}} - a_{i_{m+1}}) +$$

$$+ o(\delta_D^{(m+1)(1+\alpha)}) \quad (\delta_D(x,a) \to 0)$$

and in the case (14) the theorem is proved.

The case of boundedness of $(m+1)$-derivatives near ξ is analogous. □

Remark. Problems connecting with questions of the smooth approximation of functions near boundary points it is have many exits to tasks of the classic analysis. For example, if the boundary of the domain D is quasiconformal (see [6]), i.e. there exists a quasiconformal homeomorphism

$$T : \mathbb{R}^n \to \mathbb{R}^n, \quad T(B) = D, \quad B = \{x \in \mathbb{R}^n : |x| < 1\},$$

then smoothness of T near a boundary point implies a possibility of reduction of the initial problem of approximation near the point $a \in \partial D$ to a problem of approximation near inner point $a \in \mathbb{R}^n$. Observe, however, that the structure of D, admitting existence of m-smooth mapping T of the showed form, has not been investigated up to now.

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O LOKALNEJ APROKSYMACJI FUNKCJI W PRZSTRZENIACH ANIZOTROPOWYCH

S t r e s z c z e n i e

Dowodzimy uogólnionego twierdzenia o istnieniu różniczki zupełnej w punkcie dla funkcji w obszarach przestrzeni anizotropowych. Jako wnioski uzyskujemy pewne wyniki o aproksymacji funkcji n-wielomianami bliskimi do punktów wewnętrznych i brzegowych w podobszarach przestrzeni $\mathbb{R}^n$.

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In memory of
Professor Promarz M. Tamrazov

Alexandr Bakhtin and Iryna Denega

ADDENDUM TO A THEOREM ON EXTREMAL DECOMPOSITION OF THE COMPLEX PLANE

Summary

The paper is devoted to extremal problems of the geometric function theory of complex variable related with estimates of functionals defined on systems of non-overlapping domains. In particular, the investigation is focused on the strengthening and generalization of some known results in this theory.

Keywords and phrases: functionals on systems of non-overlapping domains, extremal decomposition of the complex plane

1. Introduction

In geometric function theory of complex variable extremal problems on non-overlapping domains are well-known classic direction and have a rich history (see [1–20]). Paper [1] was start point for this direction, in which, was first proposed and solved the problem of maximizing the product of conformal radii of two non-overlapping simply connected domains. It was the first result of this direction. Further, this problem was generalized in the many works of other authors. This article concerns one of the well-known problem of the direction.

Let $\mathbb{N}$, $\mathbb{R}$ be a set of positive integers and real numbers, respectively, $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be a Riemann sphere and $\mathbb{R}^+ = (0,\infty)$.

Let $r(B,a)$ be a inner radius of domain $B \subset \overline{\mathbb{C}}$, with respect to a point $a \in B$ (see [3], [13], [17]).

Let $n \in \mathbb{N}$. A set of points $A_n := \left\{a_k \in \mathbb{C} : \ k = \overline{1,n}\right\}$, is called *n-radial* system, if $|a_k| \in \mathbb{R}^+$, $k = \overline{1,n}$, and $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

Denote

$$\alpha_k := \frac{1}{\pi}\arg\frac{a_{k+1}}{a_k},\ \alpha_{n+1} := \alpha_1,\ k = \overline{1,n}.$$

Consider next functional

$$J_n(\gamma) = r^{\gamma}\left(B_0, 0\right)\prod_{k=1}^{n} r\left(B_k, a_k\right), \tag{1}$$

where $\gamma \in \mathbb{R}^{+}$, B_0, B_1, B_2,..., B_n, $n \geq 2$ is non-overlapping domains (e.i. $B_p \cap B_j =$ if $p \neq j$) in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, $a_k \in B_k$, $k = \overline{0,n}$ and $\gamma \leq n$.

In this paper we consider one open problem of V.N. Dubinin, which was formulated in the works [13, p. 68, no.9.2] [18, p. 381,no.16]:

Problem. (V. N. Dubinin) Prove that the maximum of functional (1) is attended for some configuration of domains which have n-tuple symmetry.

This problem has been studied in many papers (see, for example, [12, 15, 17, 18, 20]). Currently for this problem are well-known only partial results. This extremal problem related to class of extremal problems with so-called "free" poles of corresponding quadratic differentials. The fact that a quadratic differential corresponds to each extremal problem was announced by O. Teichmuller (see [4, p. 49]). By the middle of 70-ty extremal problems on non-overlapping domains were associated with quadratic differentials with fixed poles.

In 1968 P. M. Tamrazov in article [7] suggested that the study of extremal problems which correspond to quadratic differentials with free poles is very interesting and useful. In this paper P. M. Tamrazov first considered and fully solved one very important extremal problem of geometric function theory of complex variable with five simple poles. It is well known that extremal problems on non-overlapping domains corresponding quadratic differentials with poles second order. This profound idea of P. M. Tamrazov has considerably developed in the work of many scientists (see, for example, [8, 9, 13, 15–17]).

In 1974–1975 in papers of G. P. Bakhtina [8,9] this idea found an unexpected continuation in the theory of extremal problems on non-overlapping domains. Later extremal problems on non-overlapping domains considered by G. P. Bakhtina is called "extremal problems on non-overlapping domains with free poles on the circle". In 2004 A. K. Bakhtin continued development of the idea of P. M. Tamrazov and studied a number of new extremal problems on the n-radial system of points. In 1988 V. N. Dubinin studying and generalizing the formulation of extremal problems with free poles first proposed a method which allows us to solve many classes of extremal problems. V. N. Dubinin based on his method of separating transformation could solved some difficult extremal problems on non-overlapping domains with free poles on the circle. Namely, he proved next result.

Theorem 1. [12, Theorem 4] *For any different points a_k, $k = \overline{0,n}$, $(n \geqslant 2)$, which lie on the circle $|z| = 1$, and any system of non-overlapping domains D_k,*

$$a_k \in D_k \subset \overline{\mathbb{C}}, \quad k = \overline{0,n}, \quad a_0 = 0 \in D_0,$$

following inequality holds:

$$\prod_{k=0}^{n} r\left(D_k, a_k\right) \leq \frac{4^{\frac{1}{n}+n} n^n}{(n^2-1)^{\frac{1}{n}+n}} \left(\frac{n-1}{n+1}\right)^2. \tag{2}$$

If the domains D_k, $k = \overline{0,n}$, have classical Green's function then equality in (2) is attained if and only if when domains D_k and points a_k are, respectively, circular domains and poles of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-1)w^n + 1}{w^2(w^n-1)^2}\, dw^2.$$

From the method of proof of Theorem 1 immediately follows that it holds for $\gamma \in [0,1]$.

In 1996 L. V. Kovalev [15] solved the Dubinin's problem if the following conditions holds

$$0 < \alpha_k \leq 2/\sqrt{\gamma},\ k = \overline{1,n},\ n \geq 5.$$

Noted that the results of paper [15] is interesting by the method of investigation and in itself. In the case $n = 2, 3, 4$ L.V. Kovalev has not given any statements. Moreover, from the method of paper [15] follows that the result of theorem 1 holds for all $\gamma \in [0,1]$. In 2003 in paper [16] theorem 1 for $\gamma \in [0,1]$ was obtained by other method.

In monograph [17] was proposed a method of "control" functionals, which weaken the requirements on the geometry location system of points. In this way, in [17, p. 255] in 2008 it was shown that theorem 1 holds for arbitrary $\gamma \in \mathbb{R}^+$ but starting from some number $n_0(\gamma)$, $n_0 \in \mathbb{N}$. In 2009 in paper [19] it was shown that we can get more accurate results for some $\gamma > 1$ if $n \geq 5$. In 2011 Y. Zabolotnii [20] got a solution for the Dubinin's problem for $n \geq 8$ and $0 < \gamma \leq \sqrt[4]{n}$.

2. Main result

The following theorem gives additions to the result of L. V. Kovalev [15]. We shall prove

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma_2 = 1,6$, $\gamma_3 = 2,8$, and $\gamma_n = n$ if $n \geq 4$. Let $0 < \gamma \leq \gamma_n$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$, $|a_k| = 1$ such that $0 < \alpha_k \leq 2/\sqrt{\gamma}$, $k = \overline{1,n}$ and any system of non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, $a_0 = 0 \in B_0$, we have inequality*

$$r^\gamma\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leq \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \tag{3}$$

Equality in this inequality is attained when a_k and B_k, $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2.$$

Theorem 2 for $n \geq 5$ is coincides with Theorem of L.V. Kovalev. The main result consists in addition of $n = 2, 3, 4$. We prove that the Kovalev's theorem completely holds for $n \geq 4$.

Proof. Method of proof of this theorem is based on the use of separating transformation (see [12], [13]) and uses the ideas of [15, 17].

Using the methods of papers [12, 15, 17] we obtain the inequality

$$J_n(\gamma) \leq \left(\prod_{k=1}^{n} \alpha_k\right) \left[\prod_{k=1}^{n} r^{\alpha_k^2\gamma}\left(G_0^{(k)}, 0\right) \cdot r\left(G_1^{(k)}, -i\right) \cdot r\left(G_2^{(k)}, i\right)\right]^{\frac{1}{2}},$$

where $G_0^{(k)}$, $G_1^{(k)}$, $G_2^{(k)}$ are circular domains of the quadratic differential

$$Q(w)dw^2 = \frac{(4-\alpha_k^2\gamma)w^2 - \alpha_k^2\gamma}{w^2(w^2+1)^2}\, dw^2$$

$(0 \in G_0^{(k)}$, $-i \in G_1^{(k)}$, $i \in G_2^{(k)})$.

Following the works [12], [15] we introduce a function

$$P(x) = 2^{x^2+6} \cdot x^{x^2+2} \cdot (2-x)^{-\frac{1}{2}(2-x)^2} \cdot (2+x)^{-\frac{1}{2}(2+x)^2}, \quad x \in [0, 2],$$

and, as a result, next inequality is true

$$J_n(\gamma) \leq \gamma^{-\frac{n}{2}} \left[\prod_{k=1}^{n} P\left(\alpha_k\sqrt{\gamma}\right)\right]^{\frac{1}{2}}.$$

Further, we use the method suggested in works [12], [15]. Consider the extremal problem

$$\prod_{k=1}^{n} P(x_k) \longrightarrow \max; \quad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma},$$

$$x_k = \alpha_k\sqrt{\gamma}, \quad 0 < x_k \leq 2.$$

Let

$$F(x_k) = \ln\left(P(x_k)\right) \quad \text{and} \quad X^{(0)} = \left\{x_k^{(0)}\right\}_{k=1}^{n}$$

be an arbitrary extremal point. Denote

$$Z(X^{(0)}) = \sum_{k=1}^{n} F(x_k^{(0)}).$$

Repeating the arguments of paper [15] we obtain the statement: if $0 < x_k^{(0)} < x_j^{(0)} < 2$, $k \neq j$, then there is a condition of Dubinin-Kovalev-Weierstrass

$$F'(x_k^{(0)}) = F'(x_j^{(0)}),$$

and if $x_j^{(0)} = 2$, then for any $x_k^{(0)} < 2$,

$$F'(x_k^{(0)}) \leq 1,$$

where $F'(x) = 2x \ln 2x + (2-x)\ln(2-x) - (2+x)\ln(2+x) + \frac{2}{x}$ (see Fig. 1).

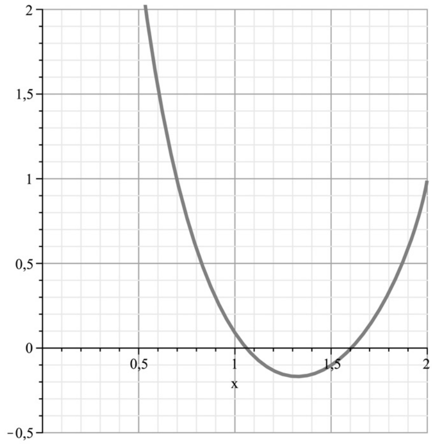

Fig. 1: Graph of a function $F'(x)$

Further, we will show that the following assertion is true: if the function

$$Z(x_1, \dots, x_n) = \sum_{k=1}^{n} F(x_k)$$

attains its maximum at the point $(x_1^{(0)}, \dots, x_n^{(0)})$ under the conditions

$$0 < x_k \leq 2, \qquad k = \overline{1, n}, \qquad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma},$$

then

$$x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}.$$

Suppose for simplicity $x_1^{(0)} \leq x_2^{(0)} \leq \ldots \leq x_n^{(0)}$. A function

$$F''(x) = \ln\left(\frac{4x^2}{4-x^2}\right) - \frac{2}{x^2}$$

is strictly increasing on $(0, 2)$ and there is x_0, $x_0 \approx 1,324661$ such that $sign F''(x) \equiv (x - x_0)$.

Consider the case $n = 2$. If $x_2^{(0)} \leq x_0$ then by the strict monotonicity $F'(x)$ on $[0, x_0]$ from the conditions of the problem we obtain that $x_1^{(0)} = x_2^{(0)}$.

Let $x_0 < x_2^{(0)} \leq 1,4$, then by increasing $F'(x)$ on $[x_0, 2]$, we have $F'(x_2^{(0)}) \leq F'(1,4) = -0,155826$.

Consider a function $\varphi(t) = (2\sqrt{t} - b)/(t-1)$, $b < 2$, $t > 1$, it decreases for $t \geq 2$, since for all $t \geq 2$

$$\varphi'(t) < (b-2)/(t-1)^2 < 0.$$

Then

$$x_1^{(0)} = 2\sqrt{\gamma} - x_2^{(0)} \leq 2\sqrt{1,6} - x_0 \leq 2\sqrt{1,6} - 1,324661 < 1,205161,$$

hence $x_1^{(0)} < 1,205161$. By decreasing $F'(x)$ on $(0, x_0)$, we have

$$F'(x_1^{(0)}) > F'(1,205161) = -0,1357233 > -0,155826 = F'(1,4),$$

which contradicts the condition of the problem.

Let $1,4 < x_2^{(0)} \leq 1,5$, then by increasing $F'(x)$ on $[x_0, 2]$, we have $F'(x_2^{(0)}) \leq F'(1,5) = -0,10207378$. Then

$$x_1^{(0)} = 2\sqrt{\gamma} - x_2^{(0)} \leq 2\sqrt{1,6} - 1,4 < 1,129822,$$

hence $x_1^{(0)} < 1,129822$. Thus

$$F'(x_1^{(0)}) > F'(1,129822) = -0,079788 > -0,10207378 = F'(1,5),$$

which contradicts the condition of the problem.

Let $1,5 < x_2^{(0)} \leq 1,6$, then by increasing $F'(x)$ on $[x_0, 2]$, we obtain $F'(x_2^{(0)}) \leq F'(1,6) = -0,005796$. Then

$$x_1^{(0)} = 2\sqrt{\gamma} - x_2^{(0)} \leq 2\sqrt{1,6} - 1,5 < 1,029822,$$

hence $x_1^{(0)} < 1,029822$. Thus

$$F'(x_1^{(0)}) > F'(1,029822) = 0,042302 > -0,005796 = F'(1,6),$$

which contradicts the condition of the problem.

Let $1,6 < x_2^{(0)} \leq 1,7$, then by increasing $F'(x)$ on $[x_0, 2]$, we have $F'(x_2^{(0)}) \leq F'(1,7) = 0,135284$. Then

$$x_1 = 2\sqrt{\gamma} - x_2^{(0)} \leq 2\sqrt{1,6} - 1,6 < 0,929822,$$

from here $x_1^{(0)} < 0,929822$. Thus

$$F'(x_1^{(0)}) > F'(0,929822) = 0,227842 > 0,135284 = F'(1,7),$$

which contradicts the condition of the problem.

Let $1,7 < x_2^{(0)} \leq 1,85$, then by increasing $F'(x)$ on $[x_0, 2]$, we obtain $F'(x_2^{(0)}) \leq F'(1,85) = 0,447263$. Then

$$x_1 = 2\sqrt{\gamma} - x_2^{(0)} \leq 2\sqrt{1,6} - 1,7 < 0,829822,$$

hence $x_1^{(0)} < 0,829822$. Thus

$$F'(x_1^{(0)}) > F'(0,829822) = 0,491217 > 0,447363 = F'(1,85),$$

which contradicts the condition of the problem.

Let $1,85 < x_2^{(0)} \leq 2$, then by increasing $F'(x)$ on $[x_0, 2]$, we have $F'(x_n^{(0)}) \leq F'(2) = 1$. Then

$$x_1 = 2\sqrt{\gamma} - x_2^{(0)} \leq 2\sqrt{1,6} - 1,85 < 0,679822,$$

from here $x_1^{(0)} < 0,679822$. Thus

$$F'(x_1^{(0)}) > F'(0,679822) = 1,084726 > 1 = F'(2),$$

which contradicts the condition of the problem. Thus for $n = 2$ theorem is proved.

Consider the case $n = 3$. If $x_3^{(0)} \leq x_0$ then by the strict monotonicity $F'(x)$ on $[0,\, x_0]$ from the conditions of the problem we obtain that $x_1^{(0)} = x_2^{(0)} = x_3^{(0)}$.

Let $x_0 < x_3^{(0)} \leq 1,65$, then $F'(x_3^{(0)}) \leq F'(1,65) = 0,058873$. And

$$\frac{1}{2}\left(x_1^{(0)} + x_2^{(0)}\right) = (2\sqrt{\gamma} - x_3^{(0)})/2 \leq$$

$$\leq (2\sqrt{2,8} - x_0)/2 \leq (2\sqrt{2,8} - 1,324661)/2 < 1,010990,$$

hence $x_1^{(0)} < 1,010990$. By decreasing $F'(x)$ on $(0,\, x_0)$, we have

$$F'(x_1^{(0)}) > F'(1,010990) = 0,072039 > 0,058873 = F'(1,65),$$

which contradicts the assumption.

Let $1,65 < x_3^{(0)} \leq 1,8$, then $F'(x_3^{(0)}) \leq F'(1,8) = 0,327581$. And

$$\frac{1}{2}\left(x_1^{(0)} + x_2^{(0)}\right) = (2\sqrt{\gamma} - x_3^{(0)})/2 \leq (2\sqrt{2,8} - 1,65)/2 < 0,848320,$$

hence $x_1^{(0)} < 0,848320$. Thus

$$F'(x_1^{(0)}) > F'(0,848320) = 0,435751 > 0,327581 = F'(1,8),$$

which contradicts the assumption.

Let $1,8 < x_3^{(0)} \leq 1,92$, then $F'(x_3^{(0)}) \leq F'(1,92) = 0,651143$. And

$$\frac{1}{2}\left(x_1^{(0)} + x_2^{(0)}\right) = (2\sqrt{\gamma} - x_3^{(0)})/2 \leq (2\sqrt{2,8} - 1,8)/2 < 0,773320,$$

hence $x_1^{(0)} < 0,773320$. Thus

$$F'(x_1^{(0)}) > F'(0,773320) = 0,682431 > 0,651143 = F'(1,92),$$

which contradicts the assumption.

Let $1,92 < x_3^{(0)} \leq 1,98$, then $F'(x_3^{(0)}) \leq F'(1,98) = 0,884285$. And

$$\frac{1}{2}\left(x_1^{(0)} + x_2^{(0)}\right) = (2\sqrt{\gamma} - x_n^{(0)})/2 \leq (2\sqrt{2,8} - 1,92)/2 < 0,713320,$$

from here $x_1^{(0)} < 0,713320$. Thus

$$F'(x_1^{(0)}) > F'(0,713320) = 0,926672 > 0,884285 = F'(1,98),$$

which contradicts the assumption.

Let $1,98 < x_3^{(0)} \leq 2$, then $F'(x_3^{(0)}) \leq F'(2) = 1$. And

$$\frac{1}{2}\left(x_1^{(0)} + x_2^{(0)}\right) = (2\sqrt{\gamma} - x_3^{(0)})/2 \leq (2\sqrt{2,8} - 1,98)/2 < 0,68332,$$

from here $x_1^{(0)} < 0,68332$. Thus

$$F'(x_1^{(0)}) > F'(0,68332) = 1,067416 > 1 = F'(2),$$

which contradicts the assumption. Thus for $n = 3$ theorem is proved.

Consider the case $n = 4$. If $x_4^{(0)} \leq x_0$ then by the strict monotonicity $F'(x)$ on $[0,\, x_0]$ from the conditions of the problem we obtain that $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)}$.

Let $x_0 < x_4^{(0)} \leq 1,75$, then $F'(x_4^{(0)}) \leq F'(1,75) = 0,224369$. And

$$(n-1)^{-1} \cdot \sum_{k=1}^{n-1} x_k^{(0)} = (2\sqrt{\gamma} - x_4^{(0)})/3 \leq (2\sqrt{4} - x_0)/3 < 0,891780,$$

hence $x_1^{(0)} < 0,891780$. By decreasing $F'(x)$ on $(0,\, x_0)$, we have

$$F'(x_1^{(0)}) > F'(0,891780) = 0,317868 > 0,224369 = F'(1,75),$$

which contradicts the assumption.

Let $1,75 < x_4^{(0)} \leq 1,95$, then by increasing $F'(x)$ on $[x_0, 2]$, we have $F'(x_4^{(0)}) \leq F'(1,95) = 0,757486$. Then

$$(n-1)^{-1} \cdot \sum_{k=1}^{n-1} x_k^{(0)} = (2\sqrt{\gamma} - x_4^{(0)})/3 \leq (2\sqrt{4} - 1,75)/3 < 0,75,$$

hence $x_1^{(0)} < 0,75$. Thus

$$F'(x_1^{(0)}) > F'(0,75) = 0,771891 > 0,757486 = F'(1,95),$$

which contradicts the assumption.

Let $1,95 < x_4^{(0)} \leq 2$, then by increasing $F'(x)$ on $[x_0, 2]$, we have $F'(x_4^{(0)}) \leq F'(2) = 1$. Then

$$(n-1)^{-1} \cdot \sum_{k=1}^{n-1} x_k^{(0)} = (2\sqrt{\gamma} - x_4^{(0)})/3 \leq (2\sqrt{4} - 1,95)/3 < 0,683333,$$

from here $x_1^{(0)} < 0,683333$. Thus

$$F'(x_1^{(0)}) > F'(0,683333) = 1,067351 > 1 = F'(2),$$

which contradicts the assumption. Thus for $n = 4$ theorem is proved.

The method of proof this theorem for $n \geqslant 5$ follows from the paper [15]. From the foregoing it follows that

$$J_n(\gamma) \leq \gamma^{-\frac{n}{2}} \left[P\left(\frac{2}{n}\sqrt{\gamma}\right)\right]^{n/2}.$$

Hence, performing simple transformations we obtain the inequality (3). The implementation of the equal sign is verified directly. Theorem is proved.

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ROZSZERZENIE TWIERDZENIA O EKSTREMALNYM ROZKŁADZIE PŁASZCZYZNY ZESPOLONEJ

S t r e s z c z e n i e
Praca jest poświęcona zagadnieniom ekstremalnym teorii funkcji analitycznych zmiennej zespolonej związanych z oszcowaniami funkcjonałów określonych na układach nie zachodzących na siebie obszarów. W szczególności, badania są zogniskowane na zaostrzeniu i uogólnieniu pewnych znanych wyników tej teorii.

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In memory of
Professor Promarz M. Tamrazov

Oksana M. Mulyava, Myroslav M. Sheremeta, and Oksana M. Sumyk

RELATION BETWEEN THE MAXIMUM MODULUS AND THE MAXIMAL TERM OF DIRICHLET SERIES IN TERMS OF A CONVERGENCE CLASS

Summary

A relation between the growth of maximum modulus and the growth of maximal term of Dirichlet series in terms of a convergence class is investigated.

Keywords and phrases: Dirichlet series, convergence class

1. Introduction

Suppose that $\Lambda = (\lambda_n)$ is a sequence of positive numbers, increasing to $+\infty$, and $S(\Lambda, A)$ is the class of Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$. For $\sigma < A$ we put

$$M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\},$$

and let

$$\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$$

be the maximal term of series (1),

$$\nu(\sigma, F) = \max\{n \geq 0 : |a_n| \exp(\sigma\lambda_n) = \mu(\sigma, F)\}$$

be its central index and

$$\varkappa_n = \frac{\ln |a_n| - \ln |a_{n+1}|}{\lambda_{n+1} - \lambda_n}.$$

By $\Omega(A)$ we denote the class of positive functions Φ, unbounded on $(-\infty, A)$, such that the derivative Φ' is continuously differentiable, positive and increasing to $+\infty$ on $(-\infty, A)$. For $\Phi \in \Omega(A)$ let φ be the function inverse to Φ' and let

$$\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}$$

b on associated with Φ in the sense of Newton. Then [1; 2, p. 30] Ψ is continuously differentiable and increasing to A on $(-\infty, A)$ and φ is continuously differentiable and increasing to A on $(0, +\infty)$.

As in [3], we say that Dirichlet series belongs to a convergence Φ-class if

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln M(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{2}$$

By Cauchy inequality from (2) it follows that

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{3}$$

The next question arises: which conditions do ensure that (3) implies (2)? Further we assume that either $A = +\infty$ or $A = 0$, because the case $A \in (-\infty, +\infty)$ can be reduced to the case $A = 0$ by substituting $s - A$ for s.

In [4] it is proved that if

$$\int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \Phi'(\sigma)}{\Phi^2(\sigma)} d\sigma < +\infty, \tag{4}$$

then for each entire ($A = +\infty$) Dirichlet series with $\lambda_n = n$ relations (2) and (3) are equivalent. We remark that from (4) it follows that $\ln \Phi'(\sigma) = o(\Phi(\sigma))$ as $\sigma \to +\infty$.

If $\Phi(\sigma) = e^{\varrho\sigma}$ ($\varrho > 0$) then from (2) we obtain the definition of a convergence class, introduced in [5]. For such convergence class in [6] the following theorem is proved.

Theorem A. *For the relations*

$$\int_{0}^{\infty} \frac{\ln M(\sigma, F)}{\exp\{\varrho\sigma\}} d\sigma < +\infty \quad \textit{and} \quad \int_{0}^{\infty} \frac{\ln \mu(\sigma, F)}{\exp\{\varrho\sigma\}} d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, +\infty)$ it is necessary and sufficient that

$$\ln n = O(\lambda_n) \qquad (n \to \infty).$$

If $\Phi(\sigma) = \sigma^p$ $(p > 1)$ for $\sigma \geq \sigma_0$ then from (2) we obtain the definition of a logarithmic convergence class, for which the following theorem is true [7].

Theorem B. *For the relations*

$$\int_1^{\infty} \sigma^{-(p+1)} \ln M(\sigma, F) d\sigma < +\infty \quad \text{and} \quad \int_1^{\infty} \sigma^{-(p+1)} \ln \mu(\sigma, F) d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, +\infty)$ it is necessary and sufficient that

$$\ln n = O(\lambda_n^{p/(p-1)}) \qquad (n \to \infty).$$

For Dirichlet series with $\sigma_a = 0$ the convergence class is defined [8] by condition (2) with $\Phi(\sigma) = |\sigma|^{-\varrho}$ $(\varrho > 0)$. The following theorem is true.

Theorem C. *For the relations*

$$\int_{-1}^{0} |\sigma|^{\varrho-1} \ln M(\sigma, F) d\sigma < +\infty \quad \text{and} \quad \int_{-1}^{0} |\sigma|^{\varrho-1} \ln \mu(\sigma, F) d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, 0)$ it is necessary and sufficient that

$$\ln \ln n = o(\ln \lambda_n) \qquad (n \to \infty).$$

Finally, if we choose in (2) $\Phi(\sigma) = e^{\varrho/|\sigma|}$ $(\varrho > 0)$ then we obtain [8] the definition of the convergence class for Dirichlet series with $\sigma_a = 0$ of finite R-order. In [9] the following theorem is proved.

Theorem D. *For the relations*

$$\int_{-1}^{0} \frac{\ln M(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty \quad \text{and} \quad \int_{-1}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, 0)$ it is necessary that

$$\ln n = O(\lambda_n / \ln^2 \lambda_n) \qquad (n \to \infty)$$

and sufficient that

$$\ln n = O(\lambda_n / \ln^q \lambda_n) \qquad (n \to \infty) \quad \text{with} \quad q > 3.$$

The aim of the present investigation is to find condition on (λ_n), under which the relations (2) and (3) are equivalent in the case when the function Φ increases rapidly enough, that is $\Phi'(\sigma)/\Phi(\sigma)$ is a nondecreasing function.

2. Sufficient condition

Let $n(t)=\sum_{\lambda_n\le t}1$ be the counting function of the sequence (λ_n). From the proof of Theorem 1 from [10] the following statement follows.

Lemma 1. *Let either $A=+\infty$ or $A=0$, $\Phi\in\Omega(A)$, and $\Phi'(\sigma)/\Phi(\sigma)$ be a function, nondecreasing on $[\sigma_0,A)$. If $\ln\mu(\sigma,F)\le\Phi(\sigma)$ for all $\sigma\in[\sigma_0,A)$ and $\ln n(t)=o(t)$ as $t\to+\infty$, then $M(\sigma,F)\le\mu(\sigma,F)n(\gamma(\sigma))+1$ for $\sigma\in[\sigma_0,A)$, where $\gamma(\sigma)=\Phi'(\Psi^{-1}(\sigma+\beta(\sigma)))$ and $\beta(\sigma)=\Phi(\sigma)/\Phi'(\Psi^{-1}(\sigma))$.*

Using Lemma 1 we prove the following theorem.

Theorem 1. *Let either $A=+\infty$ or $A=0$, $\Phi\in\Omega(A)$, $\Phi'(\sigma)/\Phi(\sigma)$ be a function, nondecreasing on $[\sigma_0,A)$, and*

$$\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2}\le H<+\infty$$

for all $\sigma\in[\sigma_0,A)$. Then, for conditions (2) and (3) to be equivalent for any function $f\in S(\Lambda,A)$ it is sufficient that

$$\int\limits_{t_0}^{\infty}\frac{\ln n(t)}{t\Phi(\Psi(\varphi(t)))}dt<+\infty. \tag{5}$$

Proof. Since

$$\Psi'(\sigma)=\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2}\quad\text{and}\quad\frac{\Phi(\Psi(\varphi(t)))}{\Phi'(\Psi(\varphi(t)))}\ge\frac{\Phi(\varphi(t))}{\Phi'(\varphi(t))}$$

we have

$$\begin{aligned}
&\varliminf_{t\to+\infty}\Phi(\Psi(\varphi(t)))\int\limits_{t}^{\infty}\frac{dx}{x\Phi(\Psi(\varphi(x)))}\\
&\ge\varliminf_{t\to+\infty}\frac{(\Phi(\Psi(\varphi(t))))^2}{t\Phi(\Psi(\varphi(t)))\Phi'(\Psi(\varphi(t)))\Psi'(\varphi(t))\varphi'(t)}=\\
&=\varliminf_{t\to+\infty}\frac{\Phi(\Psi(\varphi(t)))}{\Phi'(\Psi(\varphi(t)))}\frac{(\Phi'(\varphi(t)))^2}{\Phi''(\varphi(t))\Phi(\varphi(t))t\varphi'(t)}\\
&\ge\varliminf_{t\to+\infty}\frac{\Phi(\varphi(t))}{\Phi'(\varphi(t))}\frac{(\Phi'(\varphi(t)))^2}{\Phi''(\varphi(t))\Phi(\varphi(t))t\varphi'(t)}=1.
\end{aligned}$$

Therefore, for every $\varepsilon>0$ and all $t\ge t_0(\varepsilon)$ we obtain from (5)

$$\begin{aligned}
\varepsilon>\int\limits_{t}^{\infty}\frac{\ln n(x)}{x\Phi(\Psi(\varphi(x)))}dx&\ge\ln n(t)\int\limits_{t}^{\infty}\frac{dx}{x\Phi(\Psi(\varphi(x)))}\\
&\ge\frac{(1+o(1))\ln n(t)}{\Phi(\Psi(\varphi(t)))},\quad t\to+\infty,
\end{aligned}$$

whence it follows that

$$\ln n(t) = o(\Phi(\Psi(\varphi(t)))) \quad \text{as} \quad t \to +\infty.$$

But $\Phi(\Psi(\varphi(t))) \le \Phi(\varphi(t)) = O(t)$ as $t \to +\infty$, because $\Phi(\sigma) = O(\Phi'(\sigma))$ as $\sigma \uparrow A$. Thus, $\ln n(t) = o(t)$ as $t \to +\infty$.

From (3) it follows that $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0, A)$. Therefore, by Lemma 1 $\ln M(\sigma, F) \le \ln \mu(\sigma, F) + \ln n(\gamma(\sigma)) + o(1)$ as $\sigma \uparrow A$. Hence it follows that (3) implies (2) provided

$$I(\Lambda) = \int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln n(\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{6}$$

From the nondecrease of Φ'/Φ it follows [10] that $\sigma + \beta(\sigma) \le \Psi^{-1}(\sigma)$ and $\Phi(\sigma) \le e\Phi(\Psi(\sigma))$ for $\sigma_0 \le \sigma < A$. Therefore,

$$\begin{aligned} I(\Lambda) &\le \int_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi(\sigma)} \frac{\ln n(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))))}{\Phi(\sigma)} d\sigma \\ &\le e \int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))} \frac{\ln n(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))))}{\Phi(\Psi^{-1}(\sigma))} d\sigma = \\ &= e \int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))} \frac{\ln n(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))))}{\Phi(\Psi^{-1}(\sigma))} \frac{d\Psi^{-1}(\sigma)}{(\Psi^{-1}(\sigma))'} = \\ &= e \int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)}{\Phi(\sigma)} \frac{\ln n(\Phi'(\Psi^{-1}(\sigma)))}{\Phi(\sigma)} \Psi'(\sigma) d\sigma \\ &\le eH \int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)}{\Phi(\sigma)} \frac{\ln n(\Phi'(\Psi^{-1}(\sigma)))}{\Phi(\sigma)} d\sigma. \end{aligned}$$

Since

$$(\Psi(\varphi(x)))' = \frac{\Phi(\varphi(x))}{x^2},$$

hence and from (5) we obtain

$$\begin{aligned} I(\Lambda) &\le eH \int_{x_0}^{\infty} \frac{\Phi'(\Psi(\varphi(x)))}{\Phi(\Psi(\varphi(x)))} \frac{\ln n(x)}{\Phi(\Psi(\varphi(x)))} \frac{\Phi(\varphi(x))}{x^2} dx \le \\ &\le eH \int_{x_0}^{\infty} \frac{\Phi'(\varphi(x))}{\Phi(\varphi(x))} \frac{\ln n(x)}{\Phi(\Psi(\varphi(x)))} \frac{\Phi(\varphi(x))}{x^2} dx = \int_{x_0}^{\infty} \frac{\ln n(x)}{x\Phi(\Psi(\varphi(x)))} dx < +\infty, \end{aligned}$$

that is (6) holds, and Theorem 1 is proved.

3. Necessary condition

We need the following lemmas.

Lemma 2. [10, 11] *Suppose that γ, defined on $[0,+\infty)$, is a positive, continuous and increasing to $+\infty$ function and*

$$\varliminf_{n\to+\infty}\frac{\ln n}{\gamma(\lambda_n)}>1.$$

Then there exists a subsequence (λ_k^*) of the sequence (λ_n) such that

$$k\le\exp\{\gamma(\lambda_k^*)\}+1\quad \textit{for all}\quad k\ge1\quad \textit{and}\quad k_j\ge\exp\{\gamma(\lambda_{k_j}^*)\}$$

for an increasing sequence of (k_j) of positive integers.

Lemma 3. [3] *The relation* (3) *holds if and only if*

$$\int\limits_{\sigma_0}^{A}\frac{\lambda_{\nu(\sigma,F)}}{\Phi(\sigma)}d\sigma<+\infty. \tag{7}$$

Lemma 4. [12, p. 115] *If* $\ln n=o(\lambda_n)$ *as* $n\to\infty$ *then*

$$\sigma_a=-\varliminf_{n\to+\infty}\frac{\ln|a_n|}{\lambda_n}.$$

Using Lemmas 2–4, we prove the following theorem.

Theorem 2. *Suppose that $A=+\infty$ or $A=0$ and the function $\Phi\in\Omega(A)$ is such that*

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)}\uparrow+\infty,\qquad\frac{\Phi'(\sigma)}{\Phi^2(\sigma)}\downarrow0$$

as $\sigma\uparrow A$ and

$$\Phi(\varphi(x))\Phi'(\Phi^{-1}(x))=O(x^2)\quad \textit{as}\quad x\to+\infty.$$

Then, for relations (2) *and* (3) *to be equivalent for any function $F\in S(\Lambda,A)$, it is necessary that*

$$\ln n=O\left(\frac{\lambda_n^2}{\Phi'(\Phi^{-1}(\lambda_n))}\right),\quad n\to\infty. \tag{8}$$

Proof. At first we note that

$$\frac{x^2}{\Phi'(\Phi^{-1}(x))}\uparrow+\infty$$

and

$$\frac{x}{\Phi'(\Phi^{-1}(x))}\downarrow0\quad\text{as}\quad x\to+\infty.$$

Now we assume that the sequence Λ does not satisfy condition (8). Then there exists defined on $(0,+\infty)$, positive, continuous and slowly increasing to $+\infty$ function l such that

$$\frac{xl(x)}{\Phi'(\Phi^{-1}(x))} \to 0, \quad x \to +\infty, \tag{9}$$

and

$$\varliminf_{n\to+\infty} \frac{\Phi'(\Phi^{-1}(\lambda_n)) \ln n}{\lambda_n^2 l(\lambda_n)} > 1. \tag{10}$$

Conditions of Theorem 2 imply

$$\frac{1}{x}\Phi'\left(\Phi^{-1}\left(\frac{x^2 l(x)}{\Phi'(\Phi^{-1}(x))}\right)\right) \to +\infty$$

as $x \to +\infty$. Indeed, if

$$\Phi'\left(\Phi^{-1}\left(\frac{x_k^2 l(x_k)}{\Phi'(\Phi^{-1}(x_k))}\right)\right) \le K_0 x_k$$

for some $K_0 = \mathrm{const} \ge 1$ and a sequence (x_k) increasing to $+\infty$ then

$$l(x_k) \le K_0 \frac{\Phi(\varphi(K_0 x_k))}{K_0 x_k} \frac{\Phi'(\Phi^{-1}(x_k))}{x_k} \le K_0 \frac{\Phi(\varphi(x_k))\Phi'(\Phi^{-1}(x_k))}{x_k^2} = O(1),\ k \to \infty,$$

which is impossible. Hence it follows that there exists defined on $(0,+\infty)$, positive, continuous and increasing to $+\infty$ function L such that

$$\Phi'\left(\Phi^{-1}\left(\frac{x^2 l(x)}{\Phi'(\Phi^{-1}(x))}\right)\right) \ge xL(x), \quad x \ge x_0. \tag{11}$$

In view of (10) by Lemma 2 there exists a subsequence (λ_k^*) of the sequence (λ_n) such that

$$k \le \exp\left\{\frac{(\lambda_k^*)^2 l(\lambda_k^*)}{\Phi'(\Phi^{-1}(\lambda_k^*))}\right\} + 1$$

for all $k \ge 1$ and

$$k_j \ge \exp\left\{\frac{(\lambda_{k_j}^*)^2 l(\lambda_{k_j}^*)}{\Phi'(\Phi^{-1}(\lambda_{k_j}^*))}\right\}$$

for an increasing sequence of (k_j) of positive integers.

For $\lambda_n \notin (\lambda_k^*)$ we put $a_n = 0$ and in the obtained Dirichlet series we replace λ_k^* by λ_n. We come to Dirichlet series (1) with the exponents λ_n satisfying following conditions

$$\ln n \le \frac{\lambda_n^2 l(\lambda_n)}{\Phi'(\Phi^{-1}(\lambda_n))} + 1, \quad n \ge 1, \tag{12}$$

and

$$\ln n_j \ge \frac{\lambda_{n_j}^2 l(\lambda_{n_j})}{\Phi'(\Phi^{-1}(\lambda_{n_j}))} \tag{13}$$

for an increasing sequence of (n_j) of positive integers. We can consider that the sequence (n_j) is such that $n_{j+1} > 2n_j$ and $\lambda_{m_j} > 2\lambda_{n_j}$ for $m_j = [n_{j+1}/2]$ and all $j \geq 1$, and

$$\sum_{j=1}^{\infty} \frac{1}{L(\lambda_{n_{j+1}})} < +\infty. \tag{14}$$

Let (q_k) be a sequence, increasing to A. We put $n_0 = 0$, $a_{n_0} = 1$, $a_n = 0$ for all $n_j < n < m_j$,

$$a_{n_{j+1}} = \prod_{k=0}^{j} \exp\{-q_k(\lambda_{n_{k+1}} - \lambda_{n_k})\}, \quad j = 0, 1, \dots, \tag{15}$$

and

$$a_n = a_{n_j} \exp\{-q_j(\lambda_n - \lambda_{n_j})\}, \quad m_j \leq n < n_{j+1}, \tag{16}$$

that is we obtain the Dirichlet series

$$F^*(s) = \sum_{j=0}^{\infty} \left(a_{n_j} \exp\{s\lambda_{n_j}\} + \sum_{n=m_j}^{n_{j+1}-1} a_n \exp\{s\lambda_n\} \right). \tag{17}$$

From (15) and (16) it follows that

$$\frac{\ln a_{n_j} - \ln a_{n_{j+1}}}{\lambda_{n_{j+1}} - \lambda_{n_j}} = \frac{\ln a_{n_j} - \ln a_{m_j}}{\lambda_{m_j} - \lambda_{n_j}} = \frac{\ln a_{n_j} - \ln a_{n_j+1}}{\lambda_{n_j+1} - \lambda_{n_j}} = \\ = \frac{\ln a_n - \ln a_{n+1}}{\lambda_{n+1} - \lambda_n} = q_j, \quad m_j \leq n < n_{j+1} - 1.$$

Therefore, if $q_j \leq \sigma < q_{j+1}$ then $\nu(\sigma, F^*) = n_{j+1}$ and $\mu(\sigma, F^*) = a_{n_{j+1}} \exp\{\sigma\lambda_{n_{j+1}}\}$. Since

$$\Phi''(\sigma)\Phi(\sigma) - (\Phi'(\sigma))^2 = \Phi^2(\sigma)(\Phi'(\sigma)/\Phi(\sigma))' \geq 0,$$

hence we have

$$\int_{q_1}^{A} \frac{\lambda_{\nu(\sigma,F^*)}}{\Phi(\sigma)} d\sigma = \sum_{j=1}^{\infty} \int_{q_j}^{q_{j+1}} \frac{\lambda_{\nu(\sigma,F^*)}}{\Phi(\sigma)} d\sigma = \sum_{j=1}^{\infty} \lambda_{n_{j+1}} \int_{q_j}^{q_{j+1}} \frac{d\sigma}{\Phi(\sigma)} \leq$$

$$\leq \sum_{j=1}^{\infty} \lambda_{n_{j+1}} \int_{q_j}^{q_{j+1}} \frac{\Phi''(\sigma)d\sigma}{(\Phi'(\sigma))^2} \leq \sum_{j=1}^{\infty} \frac{\lambda_{n_{j+1}}}{\Phi'(q_j)}. \tag{18}$$

On the other hand,

$$M(q_j, F^*) \geq \sum_{n=m_j}^{n_{j+1}-1} a_n \exp\{q_j\lambda_n\} = (n_{j+1} - m_j)\mu(q_j, F^*) \geq K_1 n_{j+1}, \tag{19}$$

where $K_1 \equiv \text{const}$.

We choose

$$q_j = \Phi^{-1}\left(\frac{\lambda_{n_{j+1}}^2 l(\lambda_{n_{j+1}})}{\Phi'(\Phi^{-1}(\lambda_{n_{j+1}}))} \right).$$

Then from (19) and (13) we obtain

$$\ln\, M(q_j, F^*) \geq \ln\, n_{j+1} + \ln\, K_1 \geq \frac{\lambda^2_{n_{j+1}} l(\lambda_{n_{j+1}})}{\Phi'(\Phi^{-1}(\lambda_{n_{j+1}}))} + \ln\, K_1 = \Phi(q_j) + \ln\, K_1,$$

that is the relation (2) does not hold, because (2) implies $\ln\, M(\sigma, F) = o(\Phi(\sigma))$, $\sigma \uparrow A$.

From (18), (11) and (14) we have

$$\int\limits_{q_1}^{A} \frac{\lambda_{\nu(\sigma,F^*)}}{\Phi(\sigma)} d\sigma \leq \sum_{j=1}^{\infty} \frac{\lambda_{n_{j+1}}}{\Phi'\left(\Phi^{-1}\left(\frac{\lambda^2_{n_{j+1}} l(\lambda_{n_{j+1}})}{\Phi'(\Phi^{-1}(\lambda_{n_{j+1}}))}\right)\right)} \leq \sum_{j=1}^{\infty} \frac{1}{L(\lambda_{n_{j+1}})} < +\infty,$$

that is by Lemma 3 relation (3) holds.

Finally, we prove that $\sigma_a = A$. Since $q_k \uparrow A$ $(k \to \infty)$, from (15) we have

$$\frac{\ln\, a_{n_{j+1}}}{\lambda_{n_{j+1}}} = \frac{-\sum\limits_{k=0}^{j} q_k(\lambda_{n_{k+1}} - \lambda_{n_k})}{\sum\limits_{k=0}^{j} (\lambda_{n_{k+1}} - \lambda_{n_k})} \downarrow -A, \quad j \to \infty.$$

If $m_j \leq n < n_{j+1}$ and $A = 0$ then from (16) we obtain

$$\frac{\ln\, a_n}{\lambda_n} = \frac{\ln\, a_{n_j}}{\lambda_{n_j}} \frac{\lambda_{n_j}}{\lambda_n} - q_j + q_j \frac{\lambda_{n_j}}{\lambda_n} = o\left(\frac{\lambda_{n_j}}{\lambda_n}\right) - q_j \left(1 - \frac{\lambda_{n_j}}{\lambda_n}\right) = o(1), \quad n \to \infty,$$

and if $A = +\infty$ then

$$\frac{\ln\, a_n}{\lambda_n} \leq -q_j \left(1 - \frac{\lambda_{n_j}}{\lambda_n}\right) \leq -\frac{q_j}{2} \to -\infty, \quad j \to \infty.$$

Therefore,

$$\varliminf_{n\to+\infty} \frac{\ln\, a_n}{\lambda_n} = -A,$$

and since, in view of (12) and (9), $\ln\, n = o(\lambda_n)$ $(n \to \infty)$ by Lemma 4 we have $\sigma_a = A$.

Thus, if the sequence Λ does not satisfy condition (8) then there exists a function $F \in S(\Lambda, A)$, for which relation (2) and (3) are not equivalent. Theorem 2 is proved.

4. Remarks

Using Theorems 1 and 2, it is easy to prove the following statement, which is a slight generalization of Theorem D.

Corollary 1. *Let $\varrho > 0$. For the relations*

$$\int\limits_{-1}^{0} \frac{\ln\, M(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty \quad \textit{and} \quad \int\limits_{-1}^{0} \frac{\ln\, \mu(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty$$

to be equivalent for any function $f \in S(\Lambda, 0)$, it is sufficient that

$$\int_{t_0}^{\infty} \frac{\ln^2 t}{t^2} \ln n(t) dt < +\infty \tag{20}$$

and necessary that $\ln n(t) = O(t \ln^{-2} t)$ $(t \to +\infty)$.

Indeed, for

$$\Phi(\sigma) = \exp\left\{\frac{\varrho}{|\sigma|}\right\}$$

we have

$$\Phi'(\sigma) = \frac{\varrho}{|\sigma|^2} \exp\left\{\frac{\varrho}{|\sigma|}\right\},$$

$$|\Phi^{-1}(x)| = \frac{\varrho}{\ln x}, \quad \Phi'(\Phi^{-1}(x)) = \frac{x \ln^2 x}{\varrho}$$

and since

$$\frac{\varrho}{|\varphi(x)|^2} \exp\left\{\frac{\varrho}{|\varphi(x)|}\right\} \equiv x$$

we also have

$$|\varphi(x)| = \frac{(1 + o(1))\varrho}{\ln x}$$

and

$$\Phi(\varphi(x)) = \frac{x|\varphi(x)|^2}{\varrho} = \frac{(1 + o(1))x\varrho}{\ln^2 x} \quad \text{as} \quad x \to +\infty.$$

Therefore,

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)} \uparrow +\infty, \quad \frac{\Phi'(\sigma)}{\Phi^2(\sigma)} \downarrow 0 \quad \text{as} \quad \sigma \uparrow 0$$

and

$$\Phi(\varphi(x))\Phi'(\Phi^{-1}(x)) = (1 + o(1))x^2 \quad \text{as} \quad x \to +\infty.$$

Finally,

$$\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = 1 + \frac{2|\sigma|}{\varrho}.$$

Thus, the function

$$\Phi(\sigma) = \exp\left\{\frac{\varrho}{|\sigma|}\right\}$$

satisfies all conditions of Theorems 1 and 2. Since $\Psi(\sigma) = -|\sigma|(1 + |\sigma|/\varrho)$ we have

$$\begin{aligned}\Phi(\Psi(\sigma)) &= \exp\left\{\frac{\varrho}{|\sigma|}\left(1 + \frac{|\sigma|}{\varrho}\right)^{-1}\right\} \\ &= \exp\left\{\frac{\varrho}{|\sigma|}\left(1 - \frac{(1 + o(1))|\sigma|}{\varrho}\right)\right\} = \frac{(1 + o(1))}{e}\Phi(\sigma)\end{aligned}$$

as $\sigma \uparrow 0$ and, thus,

$$\Phi(\Psi(\varphi(x))) = \frac{(1+o(1))x\varrho}{e\ln^2 x}$$

as $x \to +\infty$, which implies equivalence of (5) and (20). Finally, since

$$\Phi'(\Phi^{-1}(x)) = \frac{x\ln^2 x}{\varrho}$$

the conditions (8) and $\ln n(t) = O(t\ln^{-2} t)$ $(t \to +\infty)$ are also equivalent. Corollary 1 is proved.

Corollary 2. *Let $p > 1$. For the relations*

$$\int\limits_{\sigma_0}^{+\infty} \frac{\sigma^{p-1}\ln M(\sigma, F)}{\exp\{\sigma^p\}} d\sigma < +\infty$$

and

$$\int\limits_{\sigma_0}^{+\infty} \frac{\sigma^{p-1}\ln \mu(\sigma, F)}{\exp\{\sigma^p\}} d\sigma < +\infty$$

to be equivalent for any function $f \in S(\Lambda, +\infty)$, it is sufficient that

$$\int\limits_{t_0}^{\infty} \frac{\ln^{(p-1)/p} t}{t^2} \ln n(t) dt < +\infty \tag{21}$$

and necessary that $\ln n(t) = O(t\ln^{-(p-1)/p} t)$ $(t \to +\infty)$.

Indeed, for

$$\Phi(\sigma) = \exp\{\sigma^p\} \qquad (\sigma \geq \sigma_0)$$

we have

$$\Phi'(\sigma) = p\sigma^{p-1}\exp\{\sigma^p\}, \quad \Phi^{-1}(x) = (\ln x)^{1/p}, \quad \Phi'(\Phi^{-1}(x)) = px(\ln x)^{(p-1)/p}$$

and since

$$p(\varphi(x))^{p-1}\exp\{(\varphi(x))^p\} \equiv x$$

we also have

$$\varphi(x) = (1+o(1))(\ln x)^{1/p}$$

and

$$\Phi(\varphi(x)) = (x/p)(\varphi(x))^{-(p-1)} = (1+o(1))(x/p)(\ln x)^{-(p-1)/p} \quad \text{as} \quad x \to +\infty.$$

Therefore,

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)} \uparrow +\infty, \quad \frac{\Phi'(\sigma)}{\Phi^2(\sigma)} \downarrow 0 \quad \text{as} \quad \sigma \to +\infty$$

and

$$\Phi(\varphi(x))\Phi'(\Phi^{-1}(x)) = (1+o(1))x^2 \quad \text{as} \quad x \to +\infty.$$

Finally,

$$\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = 1 + \frac{(p-1)}{p\sigma^p}.$$

Thus, the function $\Phi(\sigma) = \exp\{\sigma^p\}$ satisfies all conditions of Theorems 1 and 2. Since $\Psi(\sigma) = \sigma - 1/(p\sigma^{p-1})$ we have

$$\Phi(\Psi(\sigma)) = \exp\left\{\sigma^p\left(1 - \frac{1}{p\sigma^p}\right)^p\right\} = \exp\left\{\sigma^p\left(1 - \frac{1+o(1)}{\sigma^p}\right)\right\} = \frac{(1+o(1))}{e}\Phi(\sigma)$$

as $\sigma \to +\infty$ and, thus,

$$\Phi(\Psi(\varphi(x))) = \frac{1+o(1)}{e}\frac{x}{p}(\ln x)^{-(p-1)/p} \quad \text{as} \quad x \to +\infty,$$

whence it follows that (5) and (21) are equivalent. Finally, since $\Phi'(\Phi^{-1}(x)) = px(\ln x)^{(p-1)/p}$ the conditions (8) and $\ln n(t) = O(t \ln^{-(p-1)/p} t)$ $(t \to +\infty)$ are also equivalent. Corollary 2 is proved.

We remark that (21) holds provided $\ln n(t) = O(t \ln^{-\alpha} t)$ $(t \to +\infty)$ with $\alpha > (2p-1)/p$.

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ZALEŻNOŚĆ MIĘDZY NAJWIĘKSZYM MODUŁEM I NAJWIĘKSZYM WYRAZEM SZEREGU DIRICHLETA W TERMINACH KLASY ZBIEŻNOŚCI

S t r e s z c z e n i e

Zbadana została zależność między wzrostem największego modułu i wzrostem największego wyrazu szeregu Dirichleta w terminach klasy zbieżności.

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Recherches sur les déformations no. 2

pp. 107–110

In memory of
Professor Promarz M. Tamrazov

Andriy Brydun, Andriy Khrystiyanyn, and Andriy Kondratyuk

ON SIMULTANEOUS REGULAR GROWTH OF THE MODULUS AND ARGUMENT OF AN ENTIRE FUNCTION

Summary

It is proved that a simultaneous regular growth of modulus and argument of an entire function with respect to a growth function λ is possible if and only if $\lambda(r) = r^{\rho}L(r)$, $\rho > 0$, $L(r)$ is a slowly varying function.

Keywords and phrases: entire function, growth function, slowly varying function, function of moderate growth, Fourier coefficients, Cauchy-Riemann equations, proximate order, function of completely regular growth

Before we formulate and prove the main result let us give the definitions needed.

Definition 1 [1]. A positive measurable function L defined on $\mathbb{R}_+$ is called *slowly varying* (in the sense of Karamata) if $L(cr)/L(r) \to 1$ as $r \to +\infty$ uniformly with respect to c on each segment $[a, b] \subset \mathbb{R}_+$.

Definition 2. A function $\lambda(r)$ defined on $\mathbb{R}_+$ is said to be *function of moderate growth* if it is positive, continuous, non-decreasing, $\lambda(r) \to +\infty$ as $r \to +\infty$, and $\lambda(2r) \le M\lambda(r)$ for some $M > 0$ and all $r > 0$.

Let f be an entire function, $f(0) \neq 0$. Choose a value $\log f(0)$ and define

$$\log f(z) = \log f(0) + \int_0^z \frac{f'(\varsigma)}{f(\varsigma)}\, d\varsigma$$

in the complex plane with radial slits from the zeroes of f to ∞.

Our main result is the following.

Theorem A. *Let λ be a function of moderate growth. In order that the relation*

$$\left(\int_0^{2\pi} \left| \log f(re^{i\theta}) - \lambda(r) H(\theta) \right|^p \, d\theta \right)^{\frac{1}{p}} = o\left(\lambda(r)\right), \quad r \to +\infty, \tag{1}$$

hold for some entire function f, $f(0) \neq 0$, some $p \in [1, +\infty)$, and a complex-valued function H from $L_p[0, 2\pi]$ with non-constant part $Re\, H$ it is necessary and sufficient that $\lambda(r) = r^\rho L(r)$, $\rho > 0$, $L(r)$ is a slowly varying function.

In order to prove the Theorem A we give preliminary lemmas.

Let λ be a function of moderate growth. Denote

$$\lambda_1(r) = \int_0^r \frac{\lambda(t)}{t}\, dt.$$

We assume $\lambda(r)$ near the origin such that the integral exists.

Lemma 1. *If λ is a function of moderate growth, then so is λ_1, and $\lambda(r) = O\left(\lambda_1(r)\right)$, $r \to +\infty$.*

Proof. We have, changing the variable,

$$\lambda_1(2r) = \int_0^{2r} \frac{\lambda(t)}{t}\, dt = \int_0^r \frac{\lambda(2\tau)}{\tau}\, d\tau \le M \lambda_1(r).$$

Further

$$\lambda_1(er) \ge \int_r^{er} \frac{\lambda(t)}{t}\, dt \ge \lambda(r), \quad r > 0.$$

Hence, $\lambda(r) \le M^2 \lambda_1(r)$. This completes the proof.

Let $c_k(r, f)$ and $a_k(r, f)$ be the Fourier coefficients of $\log|f|$ and $\arg f = \mathrm{Im}\,(\log f)$ respectively,

$$c_k(r, f) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \log|f(re^{i\theta})|\, d\theta, \quad a_k(r, f) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \arg f(re^{i\theta})\, d\theta,$$

where $k \in \mathbb{Z}$.

Lemma 2. *Let f be an entire function, $f(0) \neq 0$, and let λ be a function of moderate growth. If for some $k \neq 0$ there exist the limits*

$$\lim_{r\to\infty} \frac{c_k(r,f)}{\lambda(r)} =: c_k, \quad \lim_{r\to\infty} \frac{a_k(r,f)}{\lambda(r)} =: a_k,$$

and $c_k \neq 0$, then $a_k \neq 0$ and $\lambda(r) = r^\rho L(r)$, $\rho > 0$, $L(r)$ is a slowly varying function.

Proof. It follows from the polar form of the Cauchy-Riemann equations applied to $\log f$ that

$$a_k(r,f) = -ik \int_0^r \frac{c_k(t,f)}{t}\, dt, \quad k \in \mathbb{Z}.$$

Therefore, $a_k\lambda(r) + o(\lambda(r)) = -ikc_k\lambda_1(r) + o\left(\lambda_1(r)\right), \quad r \to \infty$, for some $k \neq 0$. Using Lemma 1 we have

$$a_k\lambda(r) = -ikc_k\lambda_1(r) + o\left(\lambda_1(r)\right), \quad r \to \infty, \tag{2}$$

Since $c_k \neq 0$, it follows from (2) that $a_k \neq 0$. Put

$$\rho = -\frac{ik}{a_k} c_k.$$

Relation (2) implies

$$\lim_{r\to\infty} \frac{\lambda(r)}{\lambda_1(r)} = \rho. \tag{3}$$

Denote

$$\frac{\log\left(\rho\lambda_1(r)\right)}{\log r} = \rho(r);$$

using (3) and applying l'Hospital rule we obtain $\rho(r) \to \rho$ as $r \to +\infty$.

Further, we have

$$\rho'(r) r \log r = \frac{\lambda(r)}{\lambda_1(r)} - \rho(r) \to 0$$

as $r \to +\infty$. Hence $\rho(r)$ is a proximate order ([2], [3]), and

$$\lambda_1(r) = \frac{1}{\rho} r^{\rho(r)}.$$

Thus [3], $\lambda(r) = r^{\rho(r)} = r^\rho L(r)$, $L(r)$ is a slowly varying function. This finishes the proof.

Proof of Theorem A. The sufficiency was proved in [4]. Then f is a function of completely regular growth in the Levin-Pfluger sense [5], [3] and H is its indicator [4]. If the relation (1) holds for some $p \in [1, +\infty)$, then it holds for all p from $[1, +\infty)$.

In order to prove the necessity note that (1) implies the existence of the limits

$$\lim_{r\to+\infty} \frac{c_k(r,f)}{\lambda(r)} = c_k, \quad \lim_{r\to+\infty} \frac{a_k(r,f)}{\lambda(r)} = a_k, \quad k \in \mathbb{Z},$$

where c_k anda_k are the Fourier coefficients of $\operatorname{Re} H$ and $\operatorname{Im} H$ respectively. Since $\operatorname{Re} H$ is non-constant then there is $k \neq 0$ such that $c_k \neq 0$. Applying Lemma 2 we obtain $\lambda(r) = r^\rho L(r),\ \rho > 0,\ L(r)$ is a slowly varying function. The proof is complete.

Note that without the assumption "$\operatorname{Re} H$ is non-constant" the conclusion of the Theorem A is not true. Indeed, in [3] an example of entire function satisfying (1) with a real constant H and an arbitrary convex with respect to $\log r$ function $\lambda(r)$ of moderate growth was constructed.

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O JEDNOCZESNYM WZROŚCIE REGULARNYM WARTOŚCI BEZWZGLĘDNEJ I ARGUMENTU FUNKCJI CAŁKOWITEJ

S t r e s z c z e n i e

Udowodniono, że jednoczesny wzrost regularny wartości bezwzględnej i argumentu funkcji całkowitej jest możliwy wtedy i tylko wtedy, kiedy funkcja rosnąca λ ma postać $\lambda(r) = r^\rho L(r),\ \rho > 0,\ L(r)$ jest funkcją o słabej zmienności $r > 0$.

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In memory of
Professor Promarz M. Tamrazov

Olena Karupu

ON SOME PROPERTIES OF INTEGRAL MODULI OF SMOOTHNESS OF CONFORMAL MAPPINGS

Summary

Some new estimates for integral moduli of smoothness of arbitrary order for the function realizing conformal mapping between the domains bounded by the smooth Jordan curves are considered.

Keywords and phrases: conformal mapping, modulus of smoothness

1. Introduction

Let consider a simply connected domain G in the complex plane bounded by a smooth Jordan curve Γ. Let $\tau = \tau(s)$ be the angle between the tangent to Γ and the positive real axis, $s = s(w)$ be the arc length on Γ.

Let $w = \varphi(z)$ be a homeomorphism of the closed unit disk $\overline{D} = \{z : |z| \leq 1\}$ onto the closure $\overline{G}$ of the domain G, conformal in the open unit disk D. Let $z = \psi(w)$ the function inverse to the function $w = \varphi(z)$.

Kellog in 1912 proved that if $\tau = \tau(s)$ satisfies Hölder condition with index α, $0 < \alpha < 1$, then the derivative $\varphi'(e^{i\theta})$ of the function $\varphi(z)$ on ∂D satisfies Hölder condition with the same index α. Afterwards this result was generalized in works by several authors: S. E. Warshawski, J. L. Geronimus, S. J. Alper, R. N. Kovalchuk, L. I. Kolesnik.

P. M. Tamrazov [1] obtained solid reinforcement for the modulus of continuity of the function $\varphi(z)$ on $\overline{D}$. Some close problems were investigated by E. P. Dolzenko.

Results by S. E. Warshawski, R. N. Kovalchuk, L. I. Kolesnik, and the present author for moduli of smoothness of the 3rd order satisfying some additional conditions were proved applying the method due to S. E. Warshawski which is based on the introduction of additional point. However, this method contains a roughening step in the replacement of finite differences (and moduli of smoothness) of order k by finite differences (and moduli of smoothness) of the 2nd order. And as a result the less sharp inequalities obtained by means of this method do not possess any property important for applications and have essentially restricted range of applications.

In 1977 P. M. Tamrazov [2] solved the problem of estimating the finite difference smoothnesses for composite function. These results gave possibility to receive generalizations and inversations of Kellog type theorems for general moduli of smoothness of arbitrary order.

Estimates for uniform curvilinear moduli of smoothness of an arbitrary order introduced by P. M. Tamrazov [1] were obtained by the present author in [3–8] (for more details see [1], [3] and [6]).

Beside this, some estimates were established by the present author for local and integral moduli of smoothness.

Let $\omega_{k,z}(f(z),\delta)$ be a noncentralized local arithmetic modulus of smoothness of order k ($k \in \mathbb{N}$) of the function $w = f(z)$ at a point z on the curve γ. Then the integral modulus of smoothness of order k for the function $w = f(z)$ on the curve γ is introduced by the formula

$$\omega_{k,z}(f(z),\delta)_p = \left\{ \int_\gamma [\omega_{k,z}(f(z),\delta)]^p d\lambda(z) \right\}^{1/p}, \quad 1 \le p < +\infty, \quad k \in \mathbb{N},$$

where $\lambda = \lambda(z)$ is the linear Lebesgue measure on the curve.

These integral moduli are the special case of integral moduli of smoothness introduced by P. M. Tamrazov in 1977. He defined integral moduli of smoothness as averaging on arbitrary measure on the curve of the respective local moduli of smoothness.

Difference between these moduli and traditional integral moduli of smoothness, introduced as the least upper bound of averaging absolute values of finite differences, is that the operators of averaging and taking of least upper bound are applied in reverse order.

2. Estimates for integral moduli of smoothness for the derivative of the function realizing conformal mapping of the unit disk onto the Jordan domain

The following results for integral general moduli of smoothness of arbitrary order for the derivative $\varphi'(z)$ of the function $\varphi(z)$ on ∂D generalizing Kellog's theorem were earlier obtained by author.

Theorem 1. (O. W. Karupu [5]). *Let*

$$\tau(s) \in L_{pk}[0,l], \quad 1 \le p < +\infty, \quad k \in \mathbb{N}.$$

Let integral modulus $\omega_k(\tau(s),\delta)_{pk}$ of smoothness of order k for the function $\tau = \tau(s)$ satisfy the condition

$$\omega_k(\tau(s),\delta)_{pk} = O[\omega(\delta)] \ (\delta \to 0),$$

where $\omega(\delta)$ is the normal majorant such that

$$\int\limits_0^l \frac{\omega(t)}{t} dt < +\infty.$$

Then the integral modulus of smoothness of order k of the derivative $\varphi'(e^{i\theta})$ for the function $\varphi(z)$ on ∂D satisfies the condition:

$$\omega_k(\varphi'(e^{i\theta}),\delta)_p = O[\nu(\delta)] \ (\delta \to 0),$$

where

$$\nu(\delta) = \int\limits_0^l \frac{\omega(x_1)}{x_1\left(1 + (x_1/\delta)^k\right)} dx_1 +$$

$$+ \sum_{j=1}^{k-1} \sum_{r_1=1}^{j-1} \cdots \sum_{r_j=1}^{r_{j-1}-1} \delta^{k-r_1} \int\limits_0^l \cdots \int\limits_0^l x_{j+1}^{r_{j-1}} \left(1 + \int\limits_{x_{j+1}}^l \frac{\omega(y)}{y_j^{r_j+1}} dy\right)$$

$$\times \left(1 + \left(\frac{x_{j+1}}{x_j}\right)^{r_{j+1}}\right) \times \prod_{p=1}^{j} \left(1 + \int\limits_{x_p}^l \frac{\omega(t_p)}{t_p^{r_{p-1}-r_p+1}} dt_p\right) \left(1 + \left(\frac{x_p}{x_{p-1}}\right)^{r_{p-1}}\right)^{-1}$$

$$\times x_p^{r_{p-1}-r_p-1} dx_1 \ldots dx_{j+1}.$$

Corollary 1. *In a partial case when the modulus of smoothness $\omega_k(\tau(s),\delta)$ of order k for the function $\tau(s)$ satisfies Hölder condition*

$$\omega_k(\tau(s),\delta)_{pk} = O(\delta^\alpha) \ (\delta \to 0), \ 0 < \alpha < k,$$

then the modulus of smoothness $\omega_k(\varphi'(e^{i\theta}),\delta)_p$ of the same order k for the derivative $\varphi'(z)$ of the function $\varphi(z)$ on ∂D satisfies the condition

$$\omega_k(\varphi'(e^{i\theta}),\delta)_p = O(\delta^\alpha) \ (\delta \to 0)$$

with the same index α.

3. Estimates for integral moduli of smoothness for the derivative of the function realizing conformal mapping of the Jordan domain onto the unit disk

The following theorem for integral modulus of smoothness of arbitrary order for the derivative $\psi'(w)$ of the function $\psi(w)$ on the curve Γ generalizes results earlier obtained by the author for the uniform curvilinear moduli of smoothness.

Theorem 2. *Let the integral modulus of smoothness $\omega_k(\tau(s),\delta)_{pk}$ of order k ($k \in \mathbb{N}$) for the function $\tau(s)$ satisfy the condition*

$$\omega_k(\tau(s),\delta)_{pk} = O[\omega(\delta)] \ (\delta \to 0),$$

where $\omega(\delta)$ is the normal majorant satisfying

$$\int\limits_0^l \frac{\omega(t)}{t} dt < +\infty.$$

Then the nonzero derivative $\psi'(w)$ continuous on $\overline{G}$ of the function $\psi(w)$ exists and satisfies on Γ the condition

$$\omega_k(\psi'(w(s)),\delta)_p = O(\eta(\delta)) \ (\delta \to 0),$$

where

$$\eta(\delta) = \mu(\delta) + \delta^{1-k(k-1)/2} \int\limits_\delta^l \frac{\mu(y)}{y^{k+1}} dy \left(\delta^k \int\limits_\delta^l \frac{\mu(t)}{t^k} dt \right)^{\frac{k(k+1)/2-1}{k}}.$$

Proof of this theorem is similar to the proof of Theorem 1.

Corollary 2. *In a partial case when the integral modulus of smoothness $\omega_k(\tau(s),\delta)_{pk}$ of order k for the function $\tau(s)$ satisfies the Hölder condition*

$$\omega_k(\tau(s),\delta)_{pk} = O(\delta^\alpha) \ (\delta \to 0), \quad 0 < \alpha < k,$$

then the integral modulus of smoothness $\omega_k(\psi'(w(s)),\delta)_p$ of the function $\psi(w)$ satisfies the condition

$$\omega_k(\psi'(w(s)),\delta)_p = O(\delta^\alpha) \ (\delta \to 0)$$

with the same index α.

4. Estimates for integral moduli of smoothness for the derivative of the function realizing conformal mapping of the Jordan domain onto the unit disk

Let G_1 and G_2 be the simply connected domains in the complex plane, bounded by smooth Jordan curves Γ_1 and Γ_2.

Let $\tau_1(s_1)$ be the angle between the tangent to Γ_1 and the positive real axis, $s_1(\zeta)$ be the arc length on Γ_1. Let $\tau_2(s_2)$ be the angle between the tangent to Γ_2 and the positive real axis, $s_2(w)$ be the arc length on Γ_2.

Let $w = f(\zeta)$ be a homeomorphism of the closure $\overline{G_1}$ of the domain G_1 onto the closure $\overline{G_2}$ of the domain G_2, conformal in the domain G_1.

Theorem 3. *Let moduli of smoothness $\omega_k(\tau_1(s_1), \delta)_{pk}$ and $\omega_k(\tau_2(s_2), \delta)_{pk}$ of order k $(k \in \mathbb{N})$ for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy the Hölder condition*

$$\omega_k(\tau_1(s_1), \delta)_{pk} = O(\delta^\alpha) \ (\delta \to 0)$$

and

$$\omega_k(\tau_2(s_2), \delta)_{pk} = O(\delta^\alpha) \ (\delta \to 0)$$

with the same index α, $0 < \alpha < k$.

Then the integral modulus of smoothness $\omega_k(f', \delta)_p$ of the derivative of function $f(\zeta)$ on Γ_1 satisfies the Hölder condition

$$\omega_k(f', \delta)_p = O(\delta^\alpha) \ (\delta \to 0)$$

with the same index α.

Proof of this theorem is based on Corollary 1 to Theorem 1, Corollary 2 to Theorem 2 and on estimates for finite difference smoothnesses for the composite function.

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O PEWNYCH WŁASNOŚCIACH CAŁKOWYCH MODUŁÓW GŁADKOŚCI ODWZOROWAŃ KONFOREMNYCH

S t r e s z c z e n i e

W pracy uzyskano pewne nowe oszacowania całkowych modułów gładkości dowolnego rzędu dla funkcji realizującej odwzorowanie konforemne między obszarami ograniczonymi przez gładkie krzywe Jordana.

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B U L L E T I N

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In memory of
Professor Promarz M. Tamrazov

Krzysztof Pomorski and Przemysław Prokopow

NUMERICAL SOLUTIONS OF NEARLY TIME-INDEPENDENT GINZBURG-LANDAU EQUATION FOR VARIOUS SUPERCONDUCTING STRUCTURES

I. COMPUTATIONAL MODEL AND CALCULATIONS

Summary

The use of relaxation method in solving static and nearly time independent Ginzburg-Landau (GL) equations is described. The main interest is focused on the solution of GL equations applied to unconventional Josephson junction made by putting non-superconducting strip on the top of superconducting strip for s, d and p-wave superconductor. Certain solutions of Ginzburg-Landau equation are obtained in the case of placement of Josephson junction in time dependent temperature gradient, time dependent or time independent external magnetic field or when given junction is polarized by dependent or time independent superconducting current. The results of numerical calculations are related to the mesoscopic structures obtained in experiments.

Keywords and phrases: unconventional Josephson junction and device, TDGL relaxation algorithm, temperature induced Josephson junction

1. Motivation

Studying superconducting structures is important both for fundamental and applied science. There are many levels description of superconducting or superfluid phase as by use of phenomenological or microscopic models. Because of technical complication one usually starts from phenomenological level and then moves to more microscopic

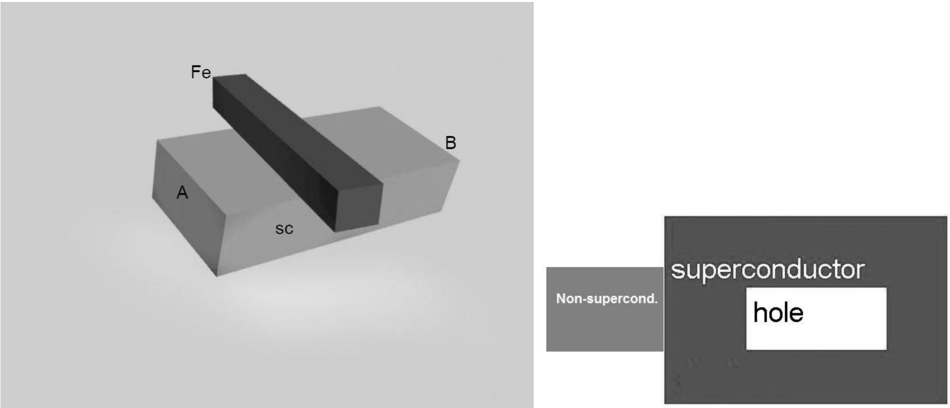

Fig. 1: Scheme of unconventional Josephson junction made by putting non-superconducting strip on the top of superconductor strip (left) and SQUID made from such structure (right). The pictures are made by Hubert Pomorski.

and fundamental description. Therefore in this paper we use Ginzburg-Landau model what is the starting point for the use of more advanced formalism as Bogoliubov-de Gennes, Usadel or Keldysh formalisms. Because of application perspectives as in THz electronics, superconducting qubit [1, 2], particular attention is paid to unconventional Josephson junction made by putting non-superconducting, ferromagnetic or ferroelectric strip on the top of superconducting strip as depicted in the Fig. 1. When the ferromagnetic or ferroelectric material is placed on the top of unconventional Josephson junction then such system is regarded as field induced Josephson junction FIJJ. If the non-superconducting strip placed on the top of superconductor is nonmagnetic we name such system as unconventional Josephson junction (uJJ).

In unconventional Josephson junction the Cooper pairs from superconductor diffuse into non-superconducting element and therefore the superconducting order parameter inside superconductor is decreased. Also unpaired electrons from non-superconducting element diffuse into superconductor what brings further reduction of superconducting order parameter. If ferromagnetic material with non-zero magnetization is placed on the top of superconductor then the magnetic field breaks the Cooper pairs and lowers more the superconducting order parameter. Having certain geometry of non-superconducting element placed on the top of thin superconductor it is possible to obtain the Josephson junction. This is because after placement of the non-superconducting element on the top of superconductor, one Cooper pair reservoir (superconductor) in terms of superconducting order parameter will be effectively separated into 2 or more superconducting reservoirs as described in [5, 4]. The interaction between reservoirs will be the origin of the Josephson effect. With such defined Josephson junction, we can build superconducting devices as the Josephson junction array, SQUID, current limiter and other elements.

2. Computational model

There are various methods, which can be used to solve the Ginzburg-Landau equations as the finite difference method, spectral methods, annealing methods (as by [18]) and many others. Because of simplicity and numerical stability even for the case of complex set of nonlinear equations the relaxation method is used. Deriving Ginzburg-Landau equations we look for the case of functional derivative of free energy functional F set to the zero with respect to the physical fields upon which it depends.

Then we obtain the following equations:

$$\frac{\delta}{\delta\psi}F[\psi,\vec{A},\vec{M},\vec{E}]=0,\frac{\delta}{\delta\vec{A}}F[\psi,\vec{A},\vec{M},\vec{E}]=0, \tag{1}$$

$$\frac{\delta}{\delta\vec{M}}F[\psi,\vec{A},\vec{M},\vec{E}]=0,\frac{\delta}{\delta\vec{E}}F[\psi,\vec{A},\vec{M},\vec{E}]=0, \tag{2}$$

where $\vec{A}$ is vector potential, $\vec{M}$ is the magnetization, ψ is the superconducting order parameter(s) and $\vec{E}$ is the electric field.

To approach the solutions given as the configuration of the $(|\psi|,\vec{M},\vec{A},\vec{E})$ fields we need to make the initial guess of physical fields configuration and order parameter in the given space using certain physical intuition. The initial guess should be not too far from the solution. Having the initial guess we perform the calculation of fields change on the given lattice with each iteration step virtual time δt according to the scheme:

$$\frac{\delta}{\delta\psi}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_1\frac{\delta\psi}{\delta t},\frac{\delta}{\delta\vec{A}}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_2\frac{\delta\vec{A}}{\delta t}, \tag{3}$$

$$\frac{\delta}{\delta\vec{M}}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_3\frac{\delta\vec{M}}{\delta t},\frac{\delta}{\delta\vec{E}}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_4\frac{\delta\vec{E}}{\delta t} \tag{4}$$

Here $\eta_1,\eta_2,\eta_3,\eta_4$ are phenomenological constants. The δt cannot have too big value since it might bring the numerical instability in the simulation. If δt has very small value the arriving to the solution is long. One of the signature of approaching the solution is the minimization of free energy functional. Then one can observe the characteristic plateua in the free energy as the function of iteration (virtual time). The instructive example of computation of superconducting order parameter distribution inside square placed in vacuum is depicted in Fig. 2. It should be underlined that the relaxation method applied here is in the framework of the Ginzburg-Landau formalism. One can use the relaxation method for other formalisms as Usadel or Eilenberger formalisms. Many applications of relaxation method as in the gauge fields are pointed by Adler [17].

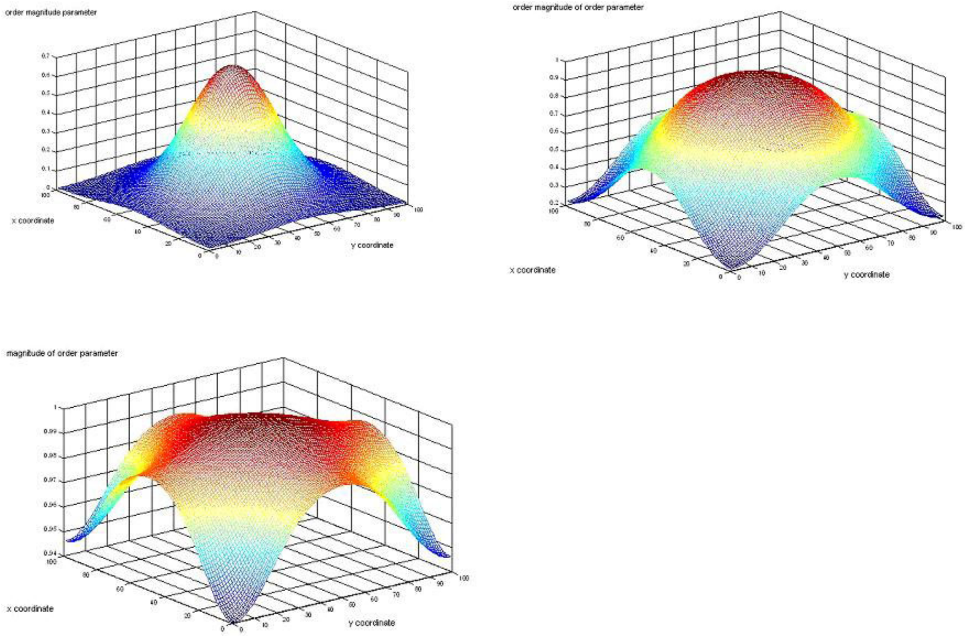

Fig. 2: Distribution of s-wave superconducing order parameter in the next iteration steps (left top, right top, left bottom) made by relaxation method for 2 dimensional superconducting square placed in vacuum, with no magnetic fields.

3. S-wave superconducting structure in time-dependent temperature gradient

For s-wave superconducting structure in time-dependent temperature gradient we can write Ginzburg-Landau equation of the following form

$$
\begin{aligned}
\gamma \frac{d}{dt}\psi(x,y,t) &= \alpha(x,y,t)\psi(x,y,t) + \beta\psi(x,y,t)|\psi(x,y,t)|^2 \\
&\quad + \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{2e}{c}A_x(x,y)\right)^2 + \left(\frac{\hbar}{i}\frac{d}{dy} - \frac{2e}{c}A_y(x,y)\right)\right. \\
&\quad \left. + \left(\frac{\hbar}{i}\frac{d}{dz} - \frac{2e}{c}A_z(x,y)\right)^2\right]\psi(x,y,t)
\end{aligned} \tag{5a}
$$

with

$$\psi(x,y,t) = |\psi(x,y,t)| \exp(i\phi(x,y,z,t)) \tag{5}$$

$$\phi = \frac{2ei}{\hbar c}\left(\int\limits_{z_a}^{z_b}(A_z(x,y,t)dz + \int\limits_{x_a}^{x_b} A_x(x,y,t)dx + \int\limits_{y_a}^{y_b} A_y(x,y,t)dy)\right) \tag{6}$$

where $\alpha(x,y,t)$ incorporates the existing gradient of temperatures and temperature field across the sample and the total electric current flowing via the sample is the sum of superconducting current and normal component of the form

$$j_z = \frac{j\hbar e^*}{2m^*}(\psi^\dagger \frac{d}{dz}\psi - h.c. - \frac{e^*}{c}A_z|\psi|^2) + \frac{dA_z}{dt}\sigma_n \tag{7}$$

$$j_y = \frac{j\hbar e^*}{2m^*}(\psi^\dagger \frac{d}{dy}\psi - h.c. - \frac{e^*}{c}A_y|\psi|^2) + \frac{dA_y}{dt}\sigma_n \tag{8}$$

$$j_x = \frac{j\hbar e^*}{2m^*}(\psi^\dagger \frac{d}{dx}\psi - h.c. - \frac{e^*}{c}A_x|\psi|^2) + \frac{dA_x}{dt}\sigma_n \tag{9}$$

The normal current component is proportional to the derivative of vector potential with time. This brings the dissipation what heats the studied system locally as it is accounted for in Drude model. It shall be underlined that the direct control of $A_z(x,y,t)$ vector potential in laboratory conditions is possible only by the control value of the integral

$$\iint j_z(x,y,t)dxdy = I(t). \tag{10}$$

We can set certain electric current value flowing via the given sample to be of the certain function of time $I(t)$. The condition of total current flowing via the system is included in the relaxation algorithm. The second controllable integral is given by external magnetic field as it can be fixed to be at the point $B(x_1, y_1, t)$, which imposes conditions on A_x and A_y. It gives another constrain $\oint \vec{A} \circ \vec{dr} = 2\pi n$, where n is the integer number.

The additionary boundary conditions comes from normal to the surface superconductor-vacuum derivatives given as

$$\begin{gathered}\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{2e}{c}A_x(x,y,t)\right)\psi(x,y,t) = 0,\\ \left(\frac{\hbar}{i}\frac{d}{dy} - \frac{2e}{c}A_y(x,y,t)\right)\psi(x,y,t) = 0\end{gathered} \tag{11}$$

and for superconductor-normal metal interface we have

$$\frac{1}{b}\psi(x,y,t) = \left(\frac{\hbar}{i}\frac{d}{dy} - \frac{2e}{c}A_y\right)\psi(x,y,t) \tag{12}$$

Let us consider the SQUID as depicted in Fig. 1. In first numerical computations we set A to be zero what means that there is no electric current flow and magnetic field in the system. We incorporate the temperature gradient into GL equations by keeping γ coefficient to be constant and by setting $\alpha(x,y,t) = \alpha_0 + a(x-x_0)(t-t_0)$. We set $t_0 = x_0 = 0$.

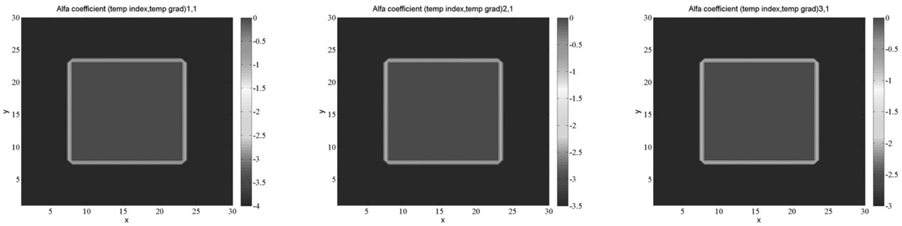

Fig. 3: $\alpha(x, y, t)$ for zero temperature gradient in times $t_0, t_0 + \Delta t, t_0 + 2\Delta t$

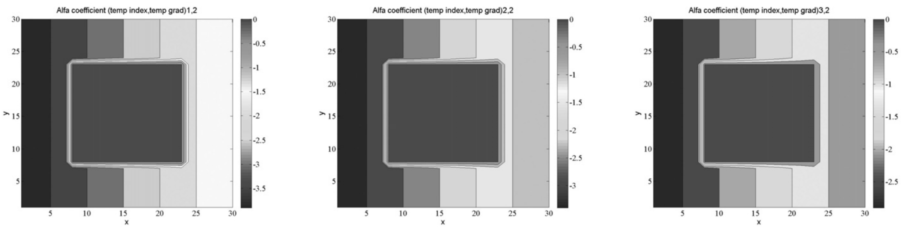

Fig. 4: $\alpha(x, y, t)$ for first temperature gradient with times $t_0, t_0 + \Delta t, t_0 + 2\Delta t$

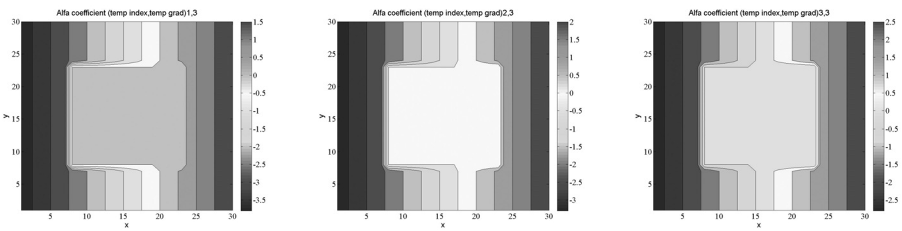

Fig. 5: $\alpha(x, y, t)$ for second temperature gradient with times $t_0, t_0 + \Delta t, t_0 + 2\Delta t$

Then we obtain the following α fields as depicted in the Fig. 3, 4 and 5. The situation when there is no temperature gradient in the sample we name as zero temperature gradient. Then temperature of sample is linearly time dependent so $\alpha(x, y, t) = \alpha_0 + a(t - t_0)$. If $\alpha(x, y, t) = \alpha_0 + a(x - x_0)(t - t_0)$ we call such situation to be first temperature gradient. In case of $\alpha(x, y, t) = \alpha_0 + 2a(x - x_0)(t - t_0)$ we name it second temperature gradient. Having given α field in dependence on time and space we can trace the time dependence of superconducting order parameter distribution in the structure.

3.1. Case of d-wave Ginzburg-Landau equation

One of the first work on study vortices in d-wave superconductor was conducted by J. Alvarez [13]. Vortex in superconductor is the example of topological defect of the superconducting order parameter induced by external magnetic field. Another type of defects in superconducting order parameter is present in the Josephson junction.

3.1.1. Testing relaxation algorithm

Ginzburg-Landau equations for d-wave superconductor in ab plane are known as GL $x^2 - y^2$ equations and are given as

$$\begin{aligned}(-\gamma_d\nabla^2 + \alpha_d)\psi_d + \gamma_v(\nabla_x^2 - \nabla_y^2)\psi_s + 2\beta_2|\psi_d|^2\psi_d + \beta_3|\psi_s|^2\psi_d + 2\beta_4\psi_s^2\psi_d^* &= 0,\\ (-\gamma_s\nabla^2 + \alpha_s)\psi_s + \gamma_v(\nabla_x^2 - \nabla_y^2)\psi_d + 2\beta_1|\psi_s|^2\psi_s + \beta_3|\psi_d|^2\psi_s + 2\beta_4\psi_d^2\psi_s^* &= 0,\end{aligned}$$

where $\gamma_\rho \equiv \hbar^2/2m_\rho$, and $\rho = d, s, v$. The current density is given by

$$\begin{aligned}\mathbf{J} &= \frac{e\hbar}{im_d}\{\psi_d^*\nabla\psi_d - \text{c.c.}\} + \frac{e\hbar}{im_s}\{\psi_s^*\nabla\psi_s - \text{c.c.}\}\\ &- \hat{x}\frac{e\hbar}{im_v}\{\psi_s^*\nabla_x\psi_d - \psi_d\nabla_x\psi_s^* - \text{c.c.}\} + \hat{y}\frac{e\hbar}{im_v}\{\psi_s^*\nabla_y\psi_d - \psi_d\nabla_y\psi_s^* - \text{c.c.}\},\end{aligned}$$

In d-wave superconductors the superconducting order parameter $\Delta(x, y, z)$ is given as

$$\Delta(x, y, z) = \psi_s(x, y, z) + \cos(2\phi)\psi_d(x, y, z) \tag{13}$$

and boundary conditiones are expressed as

$$\frac{i}{\kappa}\underline{n}(\Pi\psi_s + \frac{1}{2}(\Pi_x - \Pi_y)\psi_d) = -V_s(\psi_s), \tag{14}$$

$$\frac{i}{\kappa}\underline{n}(\Pi\psi_d + (\Pi_x - \Pi_y)\psi_s) = -V_d(\psi_d) \tag{15}$$

Here V_s and V_d depends on the material constants and $\underline{n}$ is the unit vector normal to the surface. The Δ is the global order parameter as obtained from two order parameters and $\kappa = \frac{\lambda}{\xi}$ and

$$\Pi = \frac{\hbar}{i}(\Pi_{ab} + \eta\Pi_c) - \frac{2e}{\hbar c}(A_{ab} + \eta A_c),$$

and

$$\Pi_{ab} = \underline{i}\nabla_x + \underline{j}\nabla_y, \quad A_{ab} = \underline{i}A_x + \underline{j}A_y, \quad \Pi_c = \nabla_z,$$

η-parameter accounting electron effective mass anisotropy,

$$\Pi_x = \frac{\hbar}{i}\nabla_x - \frac{2e}{\hbar c}A_x, \quad \Pi_y = \frac{\hbar}{i}\nabla_y - \frac{2e}{\hbar c}A_y.$$

4. Properties of d-wave uJJ

We have conducted the computations of the superconducting order parameter (SCOP) in 3 dimensional d-wave unconventional Josephson junction with no presence of magnetic field. The presence of vector potential and magnetic field in the system would induce additional effects in the superconducting order parameter distribution. Geometrical parametrization of the single asymmetric d-wave unconventional Josephson junction structure is given by the Fig. 16 and 2 unconventional Josephson junctions is given by Fig. 21. Numerical computation results obtained after application of relaxation method for single asymmetric uJJ of different thickness is depicted by Fig. 12–15 and for asymmetric array of 2 unconventional Josephson junction depicted by Fig. 17–20.

Experimental implementation of Magnetic Field induced Josephson junction (FIJJ) array is described by Fig. 19 and 20. The structure was produced by doctor Luis Gomez from the University of Zurich. Its modeling by Ginzburg-Landau formalism is challenging, but it is within the capacity of the relaxation method.

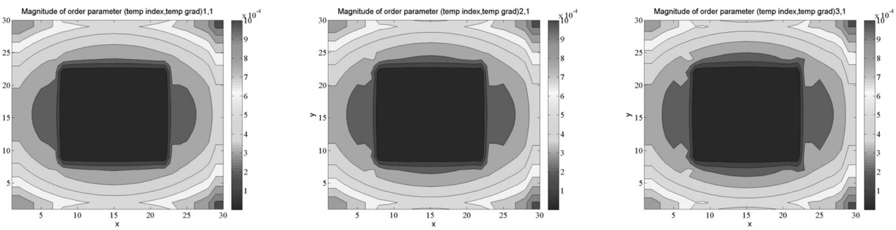

Fig. 6: Order parameter for zero temperature gradient with times $t_0, t_0 + \Delta t, t_0 + 2\Delta t$, $\gamma = 0$.

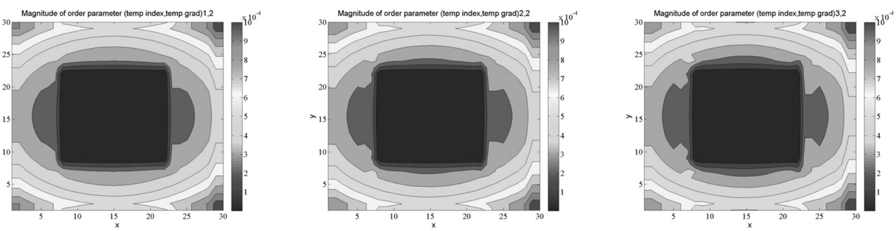

Fig. 7: Order parameter for first temperature gradient with times $t_0, t_0 + \Delta t, t_0 + 2\Delta t$, $\gamma = 0$.

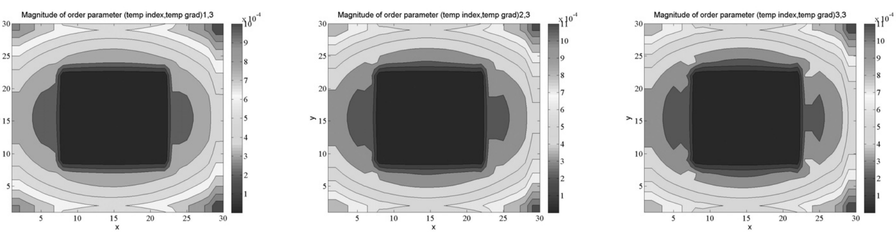

Fig. 8: Order parameter for second temperature gradient with times $t_0, t_0 + \Delta t, t_0 + 2\Delta t$, $\gamma = 0$.

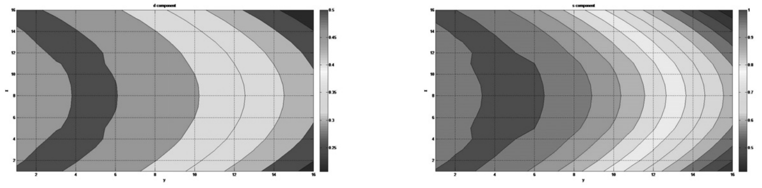

Fig. 9: D-wave uJJ, $L_z = 1.0$ left side d-wave SCOP, right side s-wave SCOP.

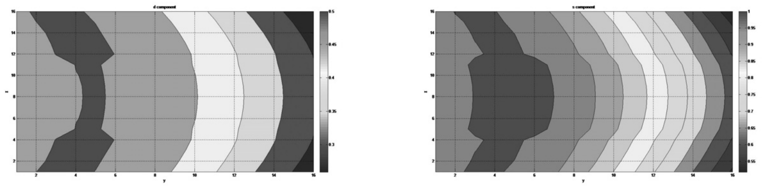

Fig. 10: D-wave uJJ, $L_z = 0.7$ left side d-wave SCOP, right side s-wave SCOP.

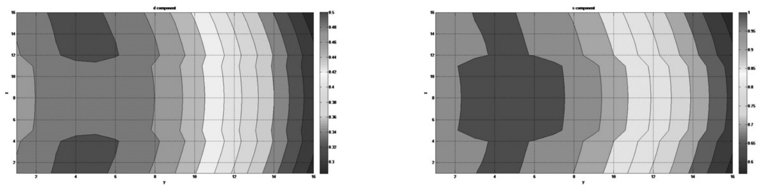

Fig. 11: D-wave uJJ, $L_z = 0.5$ left side d-wave SCOP, right side s-wave SCOP.

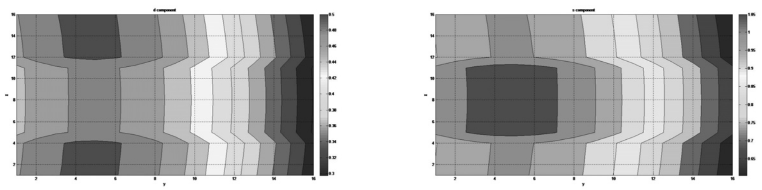

Fig. 12: D-wave uJJ, $L_z = 0.3$ left side d-wave SCOP, right side s-wave SCOP.

5. Some conclusions and further perspectives

In this work we have shown the case of numerical solutions of Ginzburg-Landau equations applied to the mesoscopic structures made of s-wave or d-wave superconductor. In case of s-wave 1uJJ SQUID the distribution of superconducting order

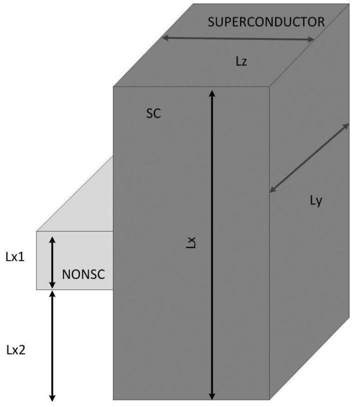

Fig. 13: Geometrical parametrization of single unconventional Josephson junction. Ly is in direction of c axis of d-wave superconductor.

parameter was quite similar to the static case as indicated by Fig. 3–8. In case of 2 d-wave uJJs as depicted in Fig. 9–18 it is not obvious whether lowering the thickness of the superconductor as in L_z direction will enhance the Josephson effect in the given structure. This is because presence of topological defect in d-wave superconductor brings lowering the d component of superconducting order parameter in the closest neighborhood and locally enhancement and later decrease the s-wave superconducting order parameter. Very good starting example to observe this effect is the work of Alvarez on vortices in d-wave superconductors. The presence of temperature griadient mishapes uniform distribution of superconducting order parameter in s-wave superconductors and in d-wave superconductors. The mechanism of this topology change is more complicated in d-wave superconductors and can be spotted by further extension of the result presented in Fig. 23. If the temperature distribution is constant around the d-wave 2 uJJs SQUID and is slowly changing the solutions of GL equations in such case for $\gamma \to 0$ can be approximated by static solutions for d-wave superconductor as reflected by Fig. 24 and Fig. 25. Typical distribution of superconducting order parameter of s-wave uJJ SQUID with no magnetic field without and with temperature gradient in x direction is depicted by Fig. 21 and Fig. 22.

All presented GL results for s-wave and d-wave superconductors have the implication of transport properties of the studied structures. This is particularly visible if we apply s or d-wave Bogoliubov-de Gennes equations using superconductor order parameter obtained from GL formalism. In very near future the conducted work will describe the experiments with d-wave uJJ matrices conducted by Luis Gomez as indicated in Fig. 19 and Fig. 20. We also believe that with use of relaxation method we can obtain the numerical results for p-wave superconductor as well. In triplet p-wave superconductors 2 superconducting order parameters coexist with non-zero magnetization.

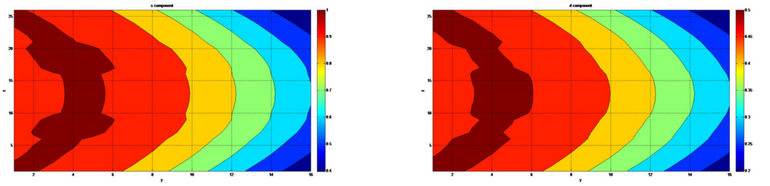

Fig. 14: D-wave 2uJJs, $L_z = 1.0$ left side s-wave SCOP, right side d-wave SCOP.

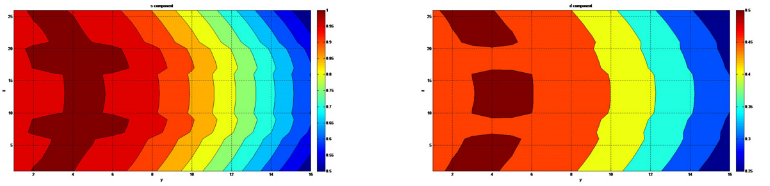

Fig. 15: D-wave 2uJJs, $L_z = 0.7$ left side s-wave SCOP, right side d-wave SCOP.

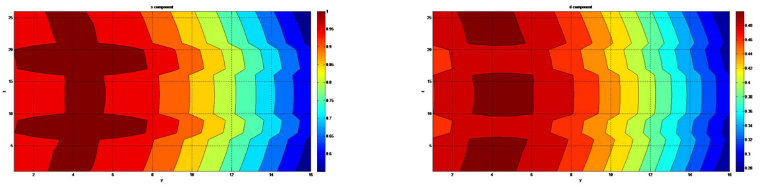

Fig. 16: D-wave 2uJJs, $L_z = 0.5$ left side s-wave SCOP, right side d-wave SCOP.

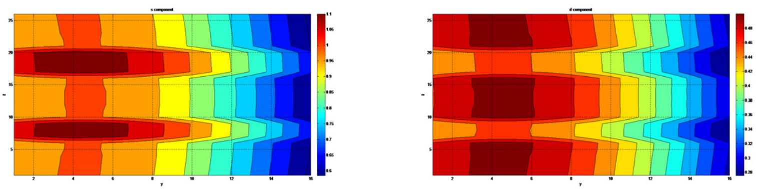

Fig. 17: SCOP distribution in D-wave 2uJJs in ab plane for $L_z = 0.3$, left side s-wave SCOP, right side d-wave SCOP.

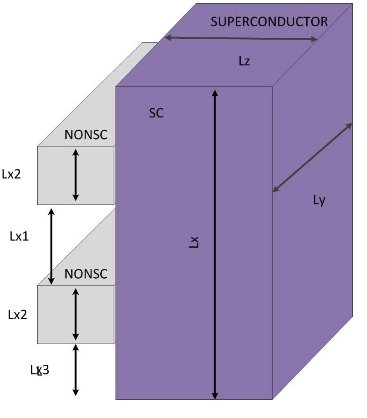

Fig. 18: Geometrical parametrization of double d-wave unconventional Josephson junction. L_y is in direction of c axis of d-wave superconductor.

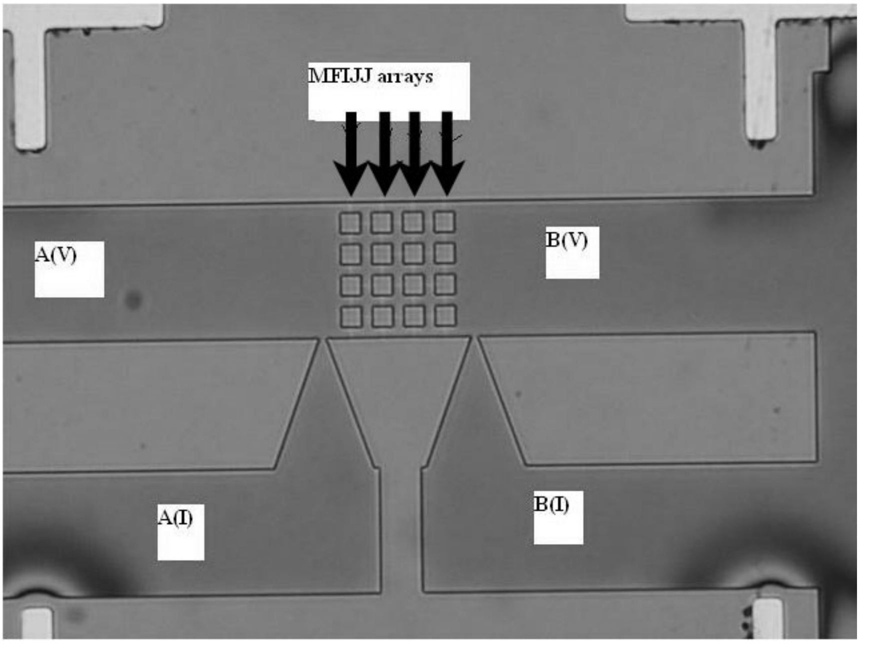

Fig. 19: Experimental implementation of 2 dimensional matrix array of uJJs implemented by putting Nb strips on the top of NbN superconducting layer. The structure was produced by doctor Luis Gomez (University of Zurich).

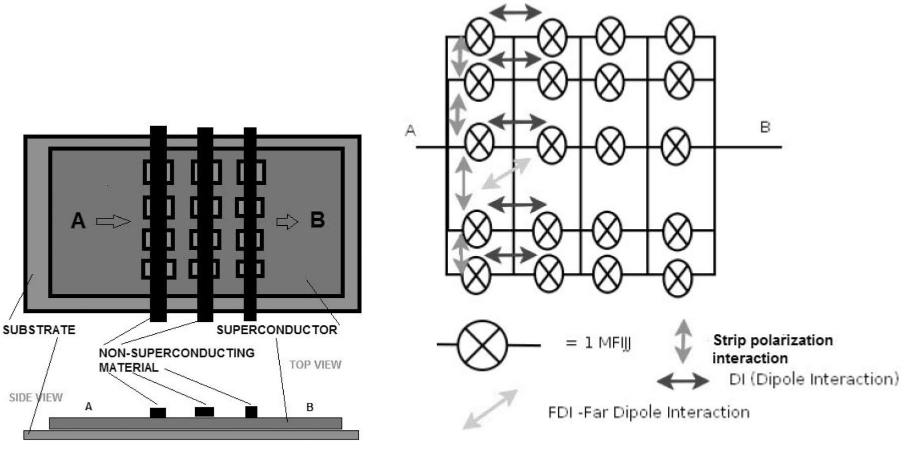

Fig. 20: Left side describes schematic view of 2 dimensional matrix array of uJJs described above. Right side describes possible interactions between Josephson junctions placed in Josephson junction matrix array. MFIJJ stands for Magnetic Field induced Josephson junction. 3 main types of interaction are specified.

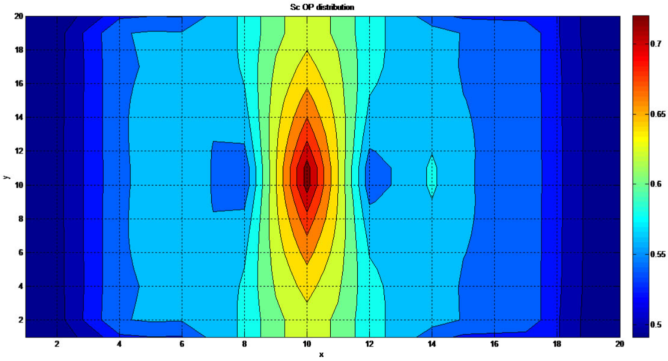

Fig. 21: SCOP distribution inside superconductor in S-wave uJJ with no presence of temperature gradient and magnetic field. The geometrical configuration of the structure is the same as depicted in Fig. 13.

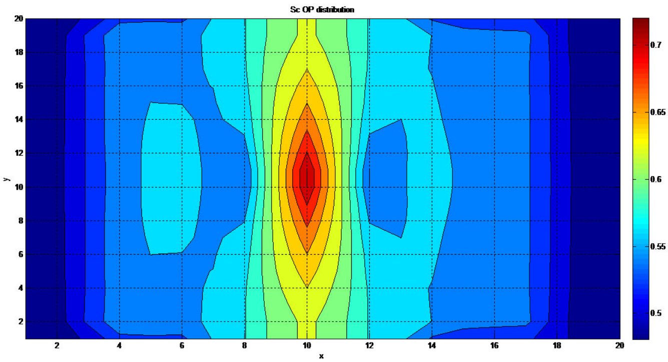

Fig. 22: SCOP distribution inside superconductor in S-wave uJJ with presence of temperature gradient in horizontal direction and no magnetic field. The geometrical configuration of the structure is the same as depicted in Fig. 13.

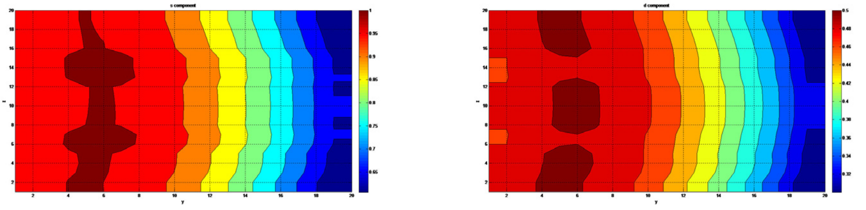

Fig. 23: ψ_s and ψ_d components of superconducting order parameter for 2 d-wave uJJs placed in uniform linearly changing in x and y temperature gradient.

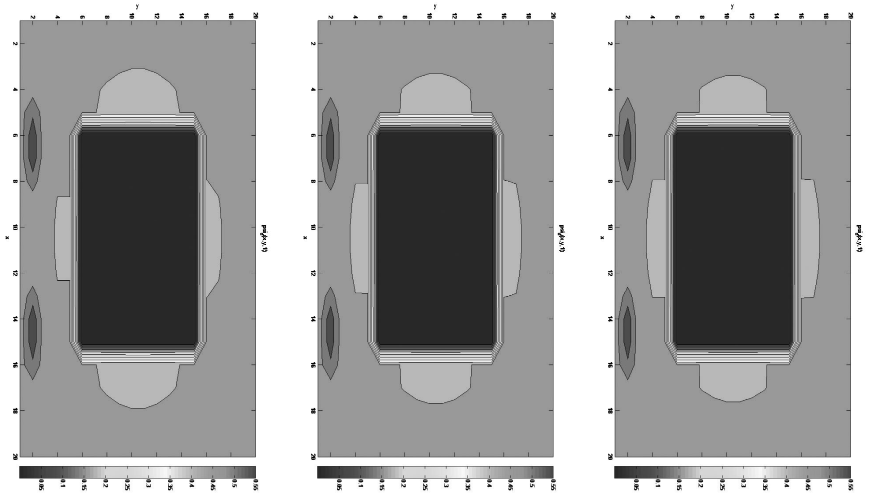

Fig. 24: $\psi_d(x, y)$ for times t_0, $t_0 + \Delta t$, $t_0 + 2\Delta t$ subjected to the uniform temperature field rising linearly with time.

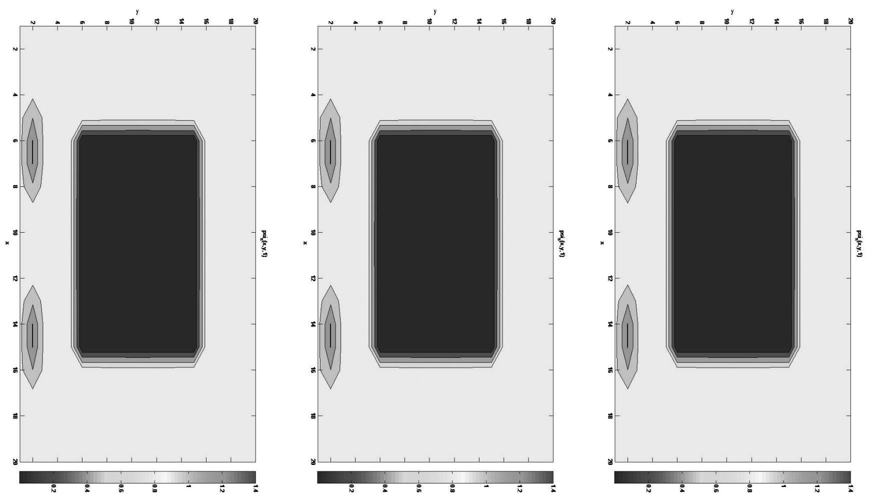

Fig. 25: $\psi_s(x, y)$ for times t_0, $t_0 + \Delta t$, $t_0 + 2\Delta t$ subjected to the uniform temperature field rising linearly with time.

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NUMERYCZNE ROZWIĄZANIA PRAWIE NIEZALEŻNYCH OD CZASU RÓWNAŃ GINZBURGA-LANDAUA DLA RÓŻNYCH NADPRZEWODZĄCYCH STRUKTUR

S t r e s z c z e n i e

Prezentujemy algorytm relaksacyjny rozwiązywania słabo zależnych od czasu równań Ginzburga-Landaua dla różnych geometrii nadprzewodzących struktur.

W szczególności uwagę koncentrujemy na niekonwencjonalnych złączach Josephsona powstałych przez nałożenie paska nienadprzewodzącego na pasek nadprzewodzący. Uzyskujemy rozkład nadprzewodzącego parametru porządku w przypadku nadprzewodnika typu s i d w sytuacji występowania gradientu temperatury. Okazuje się, że w niekonwencjonalnym złączu Josephsona zmniejszenie grubości paska nadprzewodzacego nie zawsze powoduje gładkie przykrycie nadprzewodzacych parametrów porządku w obszarze nadprzewodnika pod paskiem nadprzewodzacym. Oznacza to, że w strukturach tego typu efekt Josephsona nie zawsze jest indukowany, co jest istotne gdy chcemy implementować nadprzewodzącą elektronikę wykonujacą operację logiczne w temperaturze wyższej niż temperatura ciekłego azotu.

Rapporteurs – Referees

CONTENU DU VOLUME LXII, no. 3